Tutte's 1-Factor Theorem (West, §3.3)

Let $G = (V, E)$ be a simple graph with $n = n(G)$.

$o(G)$ = number of odd components of G (that is, components with an odd number of vertices)

— $o(G) \equiv n \pmod 2$ (i.e., $o(G)$ and $n(G)$ are either both even or both odd).

— For $S \subseteq V$

$$o(G - S) \equiv n(G - S) = n - |S| \pmod 2,$$

so

$$|S| + o(G - S) \equiv n(G) \pmod 2. \tag{1}$$

Theorem 3.3.3: (Tutte's 1-Factor Theorem) Let G be a graph. Then G has a 1-factor if and only if

$$\forall S \subset V : \quad o(G - S) \leq |S| \tag{2}$$

(this is called **Tutte's condition** for G).

Proof. Suppose that G has a perfect matching M, and let $S \subset V$. No odd component of $G - S$ has a perfect matching, so for every odd component H of $G - S$, there must exist some $v \in V(H)$ such that $w = \mathrm{sp}_M(v) \notin V(H)$. The vertex w cannot belong to any other component of $G - S$, hence must belong to S. Putting $w = f(H)$, we have a function

$$f : \{\text{odd components of } G - S\} \rightarrow S$$

that is one-to-one. In particular, $o(G - S) \leq |S|$. So we have shown that Tutte's condition is necessary for the existence of a perfect matching.

We now want to show that if G satisfies Tutte's condition, then it has a perfect matching. Note first that putting $S = \emptyset$ in (2) gives $o(G - S) = o(G) \leq |S| = 0$, so $n(G)$ is even by (1).

Claim 1: Adding an edge preserves Tutte's condition. That is, if $e \in E(H)$ and $H - e$ satisfies Tutte's condition then so does H.

To prove this, suppose that Tutte's condition holds for $H - e$. Let $S \subseteq V(H)$. If e has an endpoint in S, then $H - S = H - e - S$, so $o(H - S) = o(H - e - S) \leq |S|$. Otherwise, let J, J' be the (possibly equal) components of $H - e - S$ containing the endpoints of e. Then

$$o(H - S) = \begin{cases} o(H - e - S) & \text{if } J = J', \\ o(H - e - S) & \text{if } J \neq J' \text{ are both even}, \\ o(H - e - S) & \text{if } J \text{ is even, } J' \text{ is odd}, \\ o(H - e - S) - 2 & \text{if } J \neq J' \text{ are both odd}. \end{cases}$$

In all cases, $o(H - S) \leq o(H - e - S) \leq |S|$, proving Claim 1.

Thus, if Tutte's condition does not suffice for the existence of a 1-factor, we can choose a *maximal* counterexample G: that is, a simple graph such that

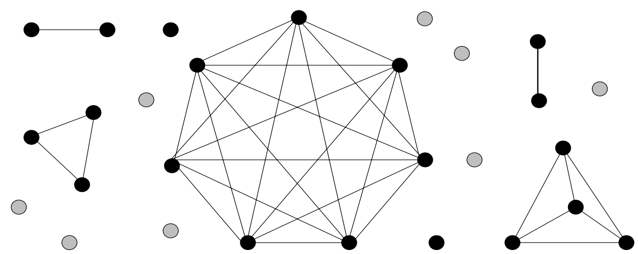
- G satisfies Tutte's condition;
- G has no 1-factor; and
- adding any single missing edge to G produces a graph with a 1-factor.

Claim 2: These conditions imply a contradiction.

The idea of the proof is to look at the graph $G - U$, where

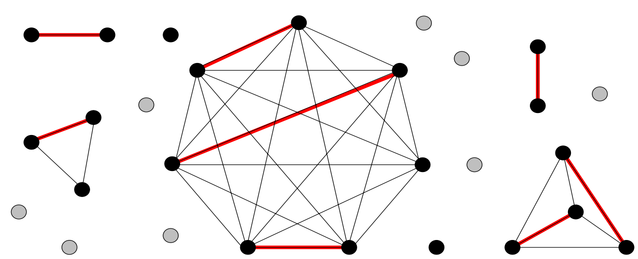
$$U = \{v \in V \mid N(v) = V - \{v\} \;=\; \{v \in V \mid d_G(v) = n - 1\}.$$

Case 1: $G - U$ is a disjoint union of cliques. For example, it might look like this:

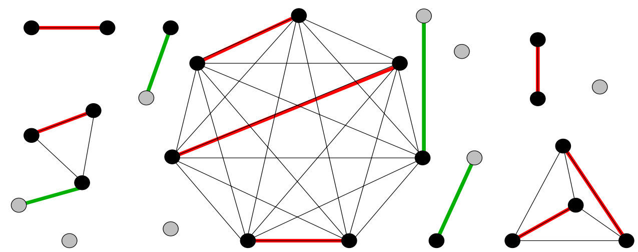


Here U consists of the vertices colored in gray. I've only drawn the edges of $G-U$; since $N(u) = V - \{u\}$ for $u \in U$, putting all the other edges in would make the picture incomprehensible. In this example, $o(G-U) = 4$ (two 1-cliques (isolated vertices), one 3-clique and one 7-clique), and $|U| = 8$ (by Tutte's condition and (1), this has the same parity as, and is greater than or equal to, $o(G - U)$).

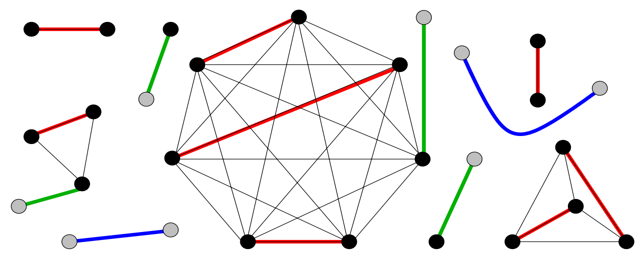
A maximum matching M on $G-U$ (the red edges in the figure below) saturates all but $o(G-U)$ vertices—every vertex of every even clique, and all vertices but one from each odd clique.



To enlarge this to a perfect matching M' of G, we first match each M-unsaturated vertex in $G - U$ to a vertex in U (the green edges).

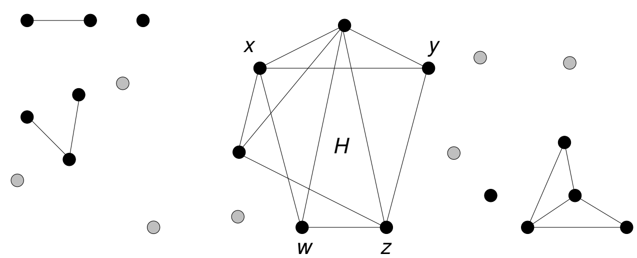


At this point, the number of unmatched vertices is $|U| - o(G - U)$. All these vertices belong to U, hence are pairwise adjacent. There is an even number of them (since $|U|$ and $o(G - U)$ have the same parity) so we can complete the perfect matching (the blue edges).

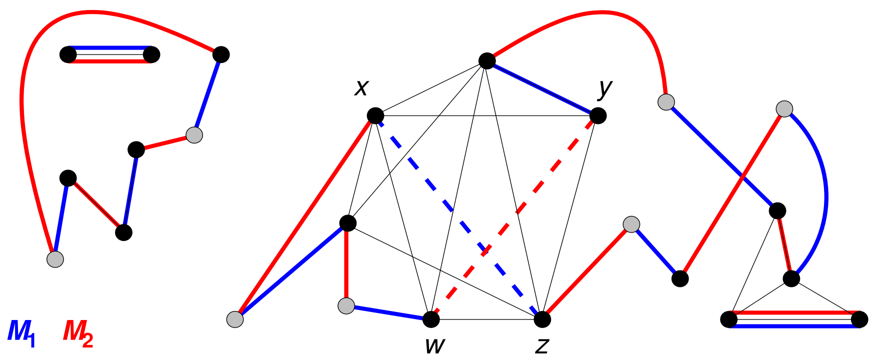


Case 2: $G - U$ is not a disjoint union of cliques.

Let H be a component of $G - U$ which is not a clique. It must have at least three vertices, and two of those vertices must be at distance 2. That is, x and z are not adjacent, but have a common neighbor y. Also, there is a vertex $w \in V(G - U)$ such that $wy \notin E$ (if no such vertex existed, then $y \in U$ by definition of U, which is not the case). (Note: w may or may not belong to H.) Again, the vertices of U are colored gray, and all edges with one or both endpoints in U are omitted.



By the choice of G, adding a single edge to G produces a graph with a perfect matching. Accordingly, let M_1 and M_2 be matchings of $G + xz$ and $G + wy$ respectively, as shown below.



The dashed edges wy and xz do not belong to G; all other edges do. Let $F = M_1 \triangle M_2$; then $xz, wy \in F$. By Lemma 3.1.9, every component of F is a path or an even cycle. Actually, each component that is a path must be an isolated vertex, otherwise its endpoints would not be saturated by both M_1 and M_2. So the component C that contains xz is an even cycle.

Case 2a: $yw \notin C$ (not the case of the example). Then

$$M_1 \triangle C \;=\; \big(M_2 \cap E(C)\big) \,\cup\, \big(M_1 - E(C)\big)$$

is a perfect matching that contains neither xz nor wy, so it is a perfect matching of G.

Case 2b: $yw \in C$. Label the vertices of C as $w,\ y,\ a_1,\ a_2,\ \ldots,\ a_p,\ z,\ x,\ b_1,\ b_2,\ b_q$. (It is possible that x and z are switched, but that case is equivalent because, we have made no distinction between these vertices—they can be interchanged.) Note also that the numbers p and q are both odd (in the example, $p = 7$ and $q = 3$). This is because the path $y, a_1, \ldots, a_p, z$ has the same number of edges in M_1 and M_2, hence has an even number of edges and an odd number of vertices. Meanwhile, $|V(C)| = 4 + p + q$ is even, so p and q have the same parity.

Now, the edge set

$$M^* = \{a_1a_2,\ \ldots,\ a_{p-2}a_{p-1},\ a_pz,\ \ yx,\ \ b_1b_2,\ \ldots,\ b_{q-2}b_{q-1},\ b_qw\} \subset E$$

(shown in green below) is a perfect matching on $V(C)$. Since $M_1 - E(C)$ (shown in yellow) is a perfect matching on $V - V(C)$, it follows that $(M_1 - E(C)) \cup M^*$ is a perfect matching of G, as desired. ■