5.5 Linear, Position-Invariant Degradations

The input-output relationship in Figure 5.1 before the restoration can be expressed as

\[
g(x, y) = H[f(x, y)] + \eta(x, y) .
\]  
(5.5-1)

First, we assume that \( \eta(x, y) = 0 \) so that \( g(x, y) = H[f(x, y)] \). \( H \) is linear if

\[
H[a f_1(x, y) + b f_2(x, y)] = a H[f_1(x, y)] + b H[f_2(x, y)] ,
\]  
(5.5-2)

where \( a \) and \( b \) are scalars and \( f_1(x, y) \) and \( f_2(x, y) \) are any two input images. If \( a = b = 1 \), then (5.5-2) becomes

\[
H[f_1(x, y) + f_2(x, y)] = H[f_1(x, y)] + H[f_2(x, y)] ,
\]  
(5.5-3)

which is called the property of additivity.

If \( f_2(x, y) = 0 \), (5.5-2) becomes

\[
H[a f_1(x, y)] = a H[f_1(x, y)] ,
\]  
(5.5-4)

which is called the property of homogeneity. It says that the response to a constant multiple of any input is equal to the response to that input multiplied by the same constant.

An operator having the input-output relationship

\[
g(x, y) = H[f(x, y)]
\]

is said to be position (space) invariant if

\[
H[f(x - \alpha, y - \beta)] = g(x - \alpha, y - \beta)
\]  
(5.5-5)

for any \( f(x, y) \) and any \( \alpha \) and \( \beta \). (5.5-5) indicates that the response at any point in the image depends only on the value of the input at that point, not on its position.
With a slight change in notation in the definition of the impulse in

\[ f(t, z) \delta(t - t_0, z - z_0) dtdz = f(t_0, z_0), \quad (4.5-3) \]

\( f(x, y) \) can be expressed as

\[ f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta) \delta(x - \alpha, y - \beta) d\alpha d\beta. \quad (5.5-6) \]

Assuming \( \eta(x, y) = 0 \), then substituting (5.5-6) into (5.5-1) we have

\[ g(x, y) = H[f(x, y)] \]

\[ = H \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta) \delta(x - \alpha, y - \beta) d\alpha d\beta \right]. \quad (5.5-7) \]

If \( H \) is a linear operator, then

\[ g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H[f(\alpha, \beta) \delta(x - \alpha, y - \beta)] d\alpha d\beta. \quad (5.5-8) \]

Since \( f(\alpha, \beta) \) is independent of \( x \) and \( y \), using the homogeneity property, it follows that

\[ g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta) H[\delta(x - \alpha, y - \beta)] d\alpha d\beta \quad (5.5-9) \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta) h(x, \alpha, y, \beta) d\alpha d\beta \quad (5.5-11) \]

where the term

\[ h(x, \alpha, y, \beta) = H[\delta(x - \alpha, y - \beta)] \quad (5.5-10) \]

is called the impulse response of \( H \).

In other words, if \( \eta(x, y) = 0 \), then \( h(x, \alpha, y, \beta) \) is the response of \( H \) to an impulse at \((x, y)\).
Equation

\[ g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta)h(x, \alpha, y, \beta) \, d\alpha \, d\beta \]  \hspace{1cm} (5.5-11)

is called the superposition (or Fredholm) integral of the first kind, and is a fundamental result at the core of linear system theory.

If \( H \) is position invariant, from

\[ H[f(x - \alpha, y - \beta)] = g(x - \alpha, y - \beta), \]  \hspace{1cm} (5.5-5)

we have

\[ H[\delta(x - \alpha, y - \beta)] = h(x - \alpha, y - \beta), \]  \hspace{1cm} (5.5-12)

and (5.5-11) reduces to

\[ g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta)h(x - \alpha, y - \beta) \, d\alpha \, d\beta. \]  \hspace{1cm} (5.5-13)

The expression (5.5-13) is the case of convolution integral

\[ f(t) \star h(t) = \int_{-\infty}^{\infty} f(\tau)h(t - \tau) \, d\tau \]  \hspace{1cm} (4.2-20)

being extended to 2-D.

Equation (5.5-13) tells us that knowing the impulse of a linear system allows us to compute its response, \( g \), to any input \( f \). The result is simply the convolution of the impulse response and the input function.

In the presence of additive noise, (5.5-11) becomes

\[ g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta)h(x, \alpha, y, \beta) \, d\alpha \, d\beta + \eta(x, y). \]  \hspace{1cm} (5.5-14)
If $H$ is position invariant, it becomes

$$g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta)h(x - \alpha, y - \beta)d\alpha d\beta + \eta(x, y). \quad (5.5-15)$$

Assuming that the values of the random noise $\eta(x, y)$ are independent of position, we have

$$g(x, y) = h(x, y) \star f(x, y) + \eta(x, y). \quad (5.5-16)$$

Based on the convolution theorem, we can express (5.5-16) in the frequency domain as

$$G(u, v) = H(u, v)F(u, v) + N(u, v). \quad (5.5-17)$$

In summary, a linear, spatially invariant degradation system with additive noise can be modeled in the spatial domain as the convolution of the degradation function with an image, followed by the additive of noise (as expressed in (5.5-16)).

The same process can be expressed in the frequency domain as stated in (5.5-17).
5.6 Estimating the Degradation Function

There are three principal ways to estimate the degradation function used in image restoration:

Estimation by Image Observation

Suppose that we are given a degraded image without any knowledge about the degradation function $H$.

Based on the assumption that the image was degraded by a linear, position-invariant process, one way to estimate $H$ is to gather information from the image itself.

In order to reduce the effect of noise, we would look for an area in which the signal content is strong.

Let the observed subimage be denoted by $g_s(x, y)$, and the processed subimage be denoted by $\hat{f}_s(x, y)$. Assuming that the effect of noise is negligible, it follows from

$$ G(u, v) = H(u, v)F(u, v) + N(u, v) $$

that

$$ H_s(u, v) = \frac{G_s(u, v)}{\hat{F}_s(u, v)} . $$

Then, we can have $H(u, v)$ based on our assumption of position invariant.

For example, suppose that a radial plot of $H_s(u, v)$ has the approximate shape of a Gaussian curve. Then we can construct a function $H(u, v)$ on a large scale, but having the same basic shape.

This estimation is a laborious process used in very specific circumstances.
Estimation by Experimentation

If equipment similar to the equipment used to acquire the degraded image is available, it is possible in principle to obtain an accurate estimate of the degradation.

Images similar to the degraded image can be acquired with various system settings until they are degraded as closely as possible to the image we wish to restore.

Then the idea is to obtain the impulse response on the degradation by imaging an impulse (small dot of light) using the same system settings.

An impulse is simulated by a bright dot of light, as bright as possible to reduce the effect of noise to negligible values. Since the Fourier transform of an impulse is a constant, it follows

\[ H(u, v) = \frac{G(u, v)}{A}. \]  \hspace{1cm} (5.6-2)

Figure 5.24 shows an example.

![Figure 5.24](image-url)
Estimation by Modeling

Degradation modeling has been used for years.

In some cases, the model can even take into account environmental conditions that cause degradations. For example, a degradation model proposed by Hufnagel and Stanley is based on the physical characteristics of atmospheric turbulence

$$H(u, v) = e^{-k(u^2 + v^2)^{5/6}},$$  \hspace{1cm} (5.6-3)

where \( k \) is a constant that depends on the nature of the turbulence.

Figure 5.25 shows examples of using (5.6-3) with different values of \( k \).
A major approach in modeling is to derive a *mathematical model* starting from basic principles.

We show this procedure by a case in which an image has been blurred by *uniform linear motion* between the image and the sensor during image acquisition.

Suppose that an image $f(x, y)$ undergoes planar motion and that $x_0(t)$ and $y_0(t)$ are the time-varying components of motion in the $x$- and $y$-directions.

The total exposure at any point of the recording medium is obtained by integrating the instantaneous exposure over the time interval when the imaging system shutter is open.

If the $T$ is the duration of the exposure, the blurred image $g(x, y)$ is

$$g(x, y) = \int_0^T f[x - x_0(t), y - y_0(t)] dt.$$  \hspace{1cm} (5.6-4)

From

$$F(\mu, \nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) e^{-j2\pi(\mu t + \nu z)} dt dz,$$  \hspace{1cm} (4.5-7)

the *Fourier transform* of (5.6-4) is

$$G(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-j2\pi(ux + vy)} dxdy$$  \hspace{1cm} (5.6-5)

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \int_0^T f[x - x_0(t), y - y_0(t)] dt \right] e^{-j2\pi(ux + vy)} dxdy$$

By reversing the order of integration,

$$G(u, v) = \int_0^T \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f[x - x_0(t), y - y_0(t)] e^{-j2\pi(ux + vy)} dxdy \right] dt$$
Since the term inside the outer brackets is the Fourier transform of the displaced function $f[x - x_0(t), y - y_0(t)]$, we have

$$G(u, v) = \int_0^T F(u, v)e^{-j2\pi[ux_0(t) + vy_0(t)]}dt$$

$$= F(u, v)\int_0^T e^{-j2\pi[ux_0(t) + vy_0(t)]}dt$$

(5.6-7)

By defining

$$H(u, v) = \int_0^T e^{-j2\pi[ux_0(t) + vy_0(t)]}dt$$

(5.6-8)

we can rewrite (5.6-7) in the familiar form

$$G(u, v) = H(u, v)F(u, v).$$

(5.6-9)

Example:

Suppose that the image in question undergoes uniform linear motion in the $x$-direction only, at a rate given by $x_0(t) = at / T$.

When $t = T$, the image has been displaced by a total distance $a$. With $y_0(t) = 0$, (5.6-8) yields

$$H(u, v) = \int_0^T e^{-j2\piux_0(t)}dt$$

$$= \int_0^T e^{-j2\piuat/T}dt$$

$$= \frac{T}{\pi ua} \sin(\pi ua)e^{-j\pi ua}$$

(5.6-10)

If we allow the $y$-component to vary as well, with the motion given by $y_0(t) = bt / T$, the degradation function becomes

$$H(u, v) = \frac{T}{\pi (ua + vb)} \sin[\pi(ua + vb)]e^{-j\pi(ua + vb)}.$$  

(5.6-11)
Example 5.10: Image blurring due to motion

Figure 5.26 (b) is an image blurred by computing the Fourier transform of the image in Figure 5.26 (a), multiplying the transform by $H(u,v)$ from (5.6-11).

The parameters used in (5.6-11) were $a = b = 0.1$ and $T = 1$. 
5.7 Inverse Filtering

The simplest approach to restoration is direct inverse filtering, where we compute an estimate, \( \hat{F}(u, v) \), of the transform of the original image by

\[
\hat{F}(u, v) = \frac{G(u, v)}{H(u, v)} \tag{5.7-1}
\]

Substituting the right side of

\[
G(u, v) = H(u, v)F(u, v) + N(u, v) \tag{5.1-2}
\]

in (5.7-1) yields

\[
\hat{F}(u, v) = F(u, v) + \frac{N(u, v)}{H(u, v)} \cdot \tag{5.7-2}
\]

The bad news is that we cannot recover the undegraded image exactly because \( N(u, v) \) is not known.

More bad news is that if the degradation function \( H(u, v) \) has zero or very small values, so the second term of (5.7-2) could easily dominate the estimate of \( \hat{F}(u, v) \).

One approach to get around the zero or small-value problem is to limit the filter frequencies to values near the origin. As discussed earlier, we know that \( H(0,0) \) is usually the highest value of \( H(u, v) \) in the frequency domain.
Example 5.11: Inverse filtering

The image in Figure 5.25 (b) was inverse filtered with

\[ \hat{F}(u, v) = \frac{G(u, v)}{H(u, v)} \quad (5.7-1) \]

using the exact inverse of the degradation function that generated that image. That is, the degradation function used was

\[ H(u, v) = e^{-k[(u-M/2)^2+(v-N/2)^2]^{5/6}} \]

with \( k = 0.0025 \). In this case, \( M = N = 480 \).

Although a Gaussian-shape function has no zeros and it is not a concern here, the degradation values become so small that the result of full inverse filtering shown in Figure 5.27 (a) is useless.
Figure 5.27 (b) through (d) show the results of cutting off values of the ratio $G(u,v)/H(u,v)$ outside a radius of 40, 70, and 85, respectively.

Values above 70 started to produce degraded images, and further increases in radius values would produce images that looked more and more like Figure 5.27 (a).

The results in Example 5.11 show the poor performance of direct inverse filtering in general.
5.8 Minimum Mean Square Error (Wiener) Filtering

Here we discuss an approach that incorporates both the degradation function and statistical characteristics of noise into the restoration process.

Considering images and noise as random variables, the objective is to find an estimate \( \hat{f} \) of the uncorrupted image \( f \) such that the mean square error between them is minimized.

The error measure is given by

\[
e^2 = E \left\{ (f - \hat{f})^2 \right\}
\]

where \( E \{ \cdot \} \) is the expected value of the argument.

By assuming that

1. the noise and the image are uncorrelated;
2. one or the other has zero mean;
3. the intensity levels in the estimate are a linear function of the levels in the degraded image.

Then, the minimum of the error function in (5.8-1) is given in the frequency domain by the expression

\[
\hat{F}(u, v) = \left[ \frac{H^*(u, v) S_f(u, v)}{S_f(u, v) |H(u, v)|^2 + S_\eta(u, v)} \right] G(u, v)
\]

\[
\hat{F}(u, v) = \left[ \frac{H^*(u, v)}{|H(u, v)|^2 + S_\eta(u, v) / S_f(u, v)} \right] G(u, v) \quad (5.8-2)
\]

\[
\hat{F}(u, v) = \left[ \frac{1}{H(u, v)} \frac{|H(u, v)|^2}{|H(u, v)|^2 + S_\eta(u, v) / S_f(u, v)} \right] G(u, v)
\]
The terms in (5.8-2) are as follows:

\[ \hat{F}(u, v) \] is the frequency domain estimate
\[ G(u, v) \] is the transform of the degraded image
\[ H(u, v) \] is the transform of the degradation function
\[ H^*(u, v) \] is complex conjugate of \( H(u, v) \)
\[ |H(u, v)|^2 = H^*(u, v)H(u, v) \]
\[ S_n(u, v) = |N(u, v)|^2 = \text{power spectrum of the noise} \]
\[ S_f(u, v) = |F(u, v)|^2 = \text{power spectrum of the undegraded image} \]

This result is known as the Wiener filter, which also is commonly referred to as the minimum mean square error filter or the least square error filter.

The Wiener filter does not have the same problem as the inverse filter with zeros in the degradation function, unless the entire denominator is zero for the same value(s) of \( u \) and \( v \).

If the noise is zero, then the Wiener filter reduces to the inverse filter.

One of the most important measures is the signal-to-noise ratio, approximated using frequency domain quantities such as

\[ SNR = \frac{\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |F(u, v)|^2}{\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |N(u, v)|^2} \]  \hspace{1cm} (5.8-3)
The mean square error given in statistical form in (5.8-1) can be approximated also in terms a summation involving the original and restored images:

\[
MSE = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} [f(x, y) - \hat{f}(x, y)]^2
\]  

(5.8-4)

If one considers the restored image to be signal and the difference between this image and the original to be noise, we can define a signal-to-noise ratio in the spatial domain as

\[
SNR = \frac{\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \hat{f}(x, y)^2}{\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} [f(x, y) - \hat{f}(x, y)]^2}
\]  

(5.8-5)

The closer \( f \) and \( \hat{f} \) are, the larger this ratio will be.

If we are dealing with white noise, the spectrum \(|N(u, v)|^2\) is a constant, which simplifies things considerably. However, \(|F(u, v)|^2\) is usually unknown.

An approach is used frequently when these quantities are not known or cannot be estimated:

\[
\hat{F}(u, v) = \left[ \frac{1}{H(u, v)} \frac{|H(u, v)|^2}{|H(u, v)|^2 + K} \right] G(u, v)
\]  

(5.8-6)

where \( K \) is a specified constant that is added to all terms of \(|H(u, v)|^2\).

Note: White noise is a random signal (or process) with a flat power spectral density. In other words, the signal contains equal power within a fixed bandwidth at any center frequency.
Example 5.12: Comparison of inverse and Wiener filtering

Figure 5.28 shows the advantage of Wiener filtering over direct inverse filtering.

![Images](image1.jpg)

**FIGURE 5.28** Comparison of inverse and Wiener filtering. (a) Result of full inverse filtering of Fig. 5.25(b). (b) Radially limited inverse filter result. (c) Wiener filter result.

**Figure 5.28 (a)** is the full inverse-filtered result from **Figure 5.27 (a)**.

**Figure 5.28 (b)** is the radially limited inverse result of **Figure 5.27 (c)**.

**Figure 5.28 (c)** shows the result obtained using

$$
\hat{F}(u, v) = \left[ \frac{1}{H(u, v)} \frac{|H(u, v)|^2}{|H(u, v)|^2 + K} \right] G(u, v)
$$

(5.8-6)

with the degradation function

$$
H(u, v) = e^{-k\left[\left(u-M/2\right)^2 + \left(v-N/2\right)^2\right]^{5/6}}
$$

used in **Example 5.11**. The value of \( K \) was chosen interactively to yield the best visual result.

By comparing **Figure 5.25 (a)** and **Figure 5.28 (c)**, we see that the **Wiener filter** yielded a result very close in appearance to the original image.
Example 5.13: Further comparisons of Wiener filtering

FIGURE 5.29 (a) 8-bit image corrupted by motion blur and additive noise. (b) Result of inverse filtering. (c) Result of Wiener filtering. (d)–(f) Same sequence, but with noise variance one order of magnitude less. (g)–(i) Same sequence, but noise variance reduced by five orders of magnitude from (a). Note in (h) how the deblurred image is quite visible through a “curtain” of noise.

5.9 Constrained Least Squares Filtering (Optional)
5.10 Geometric Mean Filter

It is possible to generalize the Wiener filter slightly to the so-called geometric mean filter:

\[
\hat{F}(u,v) = \left[ \frac{H^*(u,v)}{|H(u,v)|^2} \right]^\alpha \left[ \frac{H^*(u,v)}{|H(u,v)|^2 + \beta \left[ \frac{S_n(u,v)}{S_f(u,v)} \right]} \right]^{1-\alpha} G(u,v), \quad (5.10-1)
\]

where \( \alpha \) and \( \beta \) are positive real constants.

If \( \alpha = 1 \), this filter reduces to the inverse filter.

If \( \alpha = 0 \), the filter becomes the so-called parametric Wiener filter, which reduces to the standard Wiener filter when \( \beta = 1 \).

If \( \alpha = 1/2 \), this filter becomes a product of the two quantities raised to the same power, which is the definition of the geometric mean. When \( \beta = 1 \), the filter is also commonly referred to as the spectrum equalization filter.

With \( \beta = 1 \), as \( \alpha \) decreases below \( 1/2 \), the filter performance will tend more toward to the inverse filter; as \( \alpha \) increases above \( 1/2 \), the filter will behave more like the Wiener filter.
5.11 Image Reconstruction from Projections

In this section, we will examine the problem of reconstructing an image from a series of projections, with a focus on X-ray computed tomography (CT), which is one of the principal applications of digital image processing in medicine.

Introduction

Consider Figure 5.32 (a), which consists of a single object on a uniform background.

Suppose that we pass a thin, flat beam of X-rays from left to right, and assume that the energy of the beam is absorbed more by the object than by the background. Using a strip of X-ray absorption detectors on the other side will yield the signal, whose amplitude (intensity) is proportional to absorption.

The approach is to project the 1-D signal back across the direction from which the beam came, as Figure 5.32 (b) shows. This approach is called backprojection.
We certainly cannot determine a single object or a multitude of objects along the path of the beam by a single project.

If we rotate the position of the source-detector pair by $90\degree$ and repeat the previous procedure, we will get a backprojection image shown in Figure 5.32 (d). Adding this result to Figure 5.32 (b) will result an image illustrated in Figure 5.32 (e).

We should be able to learn more about the shape of the object in question by taking more views in the same manner, as shown in Figure 5.33.

As the number of projections increases, the strength of non-intersecting backprojects deceases relative to the strength of regions in which multiple backprojects intersect.

Figure 5.33 (f) shows the result formed from 32 projections.

The reconstructed image seems to be a reasonably good approximation to the shape of the original object. However, the image is blurred by a “halo” effect, which shows a “star” in Figure 5.33 (e). As the number of views increases, the shape of the “halo” becomes circular, as shown in Figure 5.33 (e).
Blurring in CT reconstruction is an important issue and will be addressed in later discussion.

Since the projections $180^\circ$ apart are mirror images of each other, we only need to consider angle increments halfway around a circle in order to generate all the projects required for reconstruction.

Example 5.16: Backprojection of a simple planar region containing two objects

![Example 5.16: Backprojection of a simple planar region containing two objects](image)

**FIGURE 5.34** (a) A region with two objects. (b)–(d) Reconstruction using 1, 2, and 4 backprojections $45^\circ$ apart. (e) Reconstruction with 32 backprojections $5.625^\circ$ apart. (f) Reconstruction with 64 backprojections $2.8125^\circ$ apart.