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Self-Complementary Hypergraphs

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Abstract

In this thesis, we survey the current research into self-complementary hypergraphs, and present several new results.

We characterize the cycle type of the permutations on n elements with order equal to a power of 2 which are k-complementing. The k-complementing permutations map the edges of a k-uniform hypergraph to the edges of its complement. This yields a test to determine whether a finite permutation is a k-complementing permutation, and an algorithm for generating all self-complementary k-uniform hypergraphs of order n, up to isomorphism, for feasible n. We also obtain an alternative description of the known necessary and sufficient conditions on the order of a self-complementary k-uniform hypergraph in terms of the binary representation of k.

We examine the orders of t-subset-regular self-complementary uniform hypergraphs. These form examples of large sets of two isomorphic t-designs. We restate the known necessary conditions on the order of these structures in terms of the binary representation of the rank k, and we construct 1-subset-regular self-complementary uniform hypergraphs to prove that these necessary conditions are sufficient for all ranks k in the case where $t = 1$.

We construct vertex transitive self-complementary k-hypergraphs of order n for all integers n which satisfy the known necessary conditions due to Potočnik and Šajna, and consequently prove that these necessary conditions are also sufficient. We also generalize Potočnik and Šajna's necessary conditions on the order of a vertex

transitive self-complementary uniform hypergraph for certain ranks k to give necessary conditions on the order of these structures when they are t-fold-transitive. In addition, we use Burnside's characterization of transitive groups of prime degree to determine the group of automorphisms and antimorphisms of certain vertex transitive self-complementary k-uniform hypergraphs of prime order, and we present an algorithm to generate all such hypergraphs.

Finally, we examine the orders of self-complementary non-uniform hypergraphs, including the cases where these structures are t-subset-regular or t-fold-transitive. We find necessary conditions on the order of these structures, and we present constructions to show that in certain cases these necessary conditions are sufficient.

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Part I

Self-complementary uniform hypergraphs

Chapter 1

Introduction

1.1 Definitions

For a finite set V and a positive integer k, let $V^{(k)}$ denote the set of all k-subsets of V. A *hypergraph* with vertex set V and edge set E is a pair (V, E), in which V is a finite set and E is a collection of subsets of V. A hypergraph (V, E) is called *k-uniform* (or a *k-hypergraph*) if E is a subset of $V^{(k)}$. The parameters k and $|V|$ are called the *rank* and the *order* of the k-hypergraph, respectively. The vertex set and the edge set of a hypergraph X will often be denoted by $V(X)$ and $E(X)$, respectively. Note that a 2-hypergraph is a *graph*.

An *isomorphism* between k-hypergraphs X and X' is a bijection $\phi : V(X) \to V(X')$ which induces a bijection from $E(X)$ to $E(X')$. If such an isomorphism exists, the hypergraphs X and X' are said to be *isomorphic*. An *automorphism* of X is an isomorphism from X to X. The set of all automorphisms of X will be denoted by $\mathrm{Aut}(X)$. Clearly, $\mathrm{Aut}(X)$ is a subgroup of $Sym(V(X))$, the symmetric group of permutations on $V(X)$.

The *complement* X^C of a k-hypergraph $X = (V, E)$ is the hypergraph with vertex set V and edge set $E^C = V^{(k)} \setminus E$. A k-hypergraph X is called *self-complementary* if it is isomorphic to its complement. An isomorphism between a self-complementary k-hypergraph $X = (V, E)$ and its complement X^C is called an *antimorphism* of X. The set of all antimorphisms of X will be denoted by $\mathrm{Ant}(X)$. It is easy to check that $\mathrm{Aut}(X) \cup \mathrm{Ant}(X)$ is a subgroup of $Sym(V)$, and that $\mathrm{Aut}(X)$ is an index-2 subgroup of $\mathrm{Aut}(X) \cup \mathrm{Ant}(X)$. Also, it is clear that $\mathrm{Aut}(X) = \mathrm{Aut}(X^C)$ when X is self-complementary. An antimorphism of a self-complementary k-hypergraph is often called a *k-complementing permutation*.

Let $X = (V, E)$ be a k-hypergraph, let t be a positive integer, $t < k$, and let $f \in V^{(t)}$. We define the *t-valency $val_X^t(f)$* of f in X to be the number of edges $e \in E$ containing f. A k-hypergraph X is called *t-subset-regular* if the t-valency of f in X is independent of the choice of $f \in V^{(t)}$, and hence is called the *t-valency of* X without ambiguity. A k-hypergraph is *regular* if it is 1-subset-regular. A k-hypergraph is called *t-fold-transitive*, or *t-transitive*, if $\mathrm{Aut}(X)$ acts transitively on the set of ordered t-tuples of pairwise distinct vertices of X. Clearly, every t-transitive k-hypergraph is t-subset-regular. A k-hypergraph X is called *vertex transitive* (or simple *transitive*) if it is 1-fold-transitive, and it is called *doubly transitive* if it is 2-fold-transitive. Note that for graphs (2-hypergraphs), the properties of t-subset-regularity and t-fold-transitivity are undefined unless $t \leq 2$. Doubly transitive or 2-subset-regular graphs must be complete or edgeless, and so for graphs these properties are only interesting when $t = 1$, in which case these concepts correspond to the well studied properties of regularity and vertex transitivity in graphs.

There is a connection between t-subset-regular hypergraphs and designs. Hence results from design theory are applicable to these hypergraphs and vice versa. For $t \leq k \leq n$, a *t-(n, k, λ) design* is a pair $(V, \mathcal{B})$ in which V is a set of cardinality n and $\mathcal{B}$ is a collection of k-subsets of the point set V such that every t-subset of V is contained in exactly λ elements of $\mathcal{B}$. Hence a t-subset-regular k-hypergraph X of order n is a t-(n, k, λ) design in which λ is equal to the t-valency of X. A *large set of t-(n, k, λ) designs* of size N, denoted by $LS[N](t, k, n)$, is a partition of the complete design

$(V, V^{(k)})$ into N disjoint t-(n, k, λ) designs, where $\lambda = \binom{n-t}{k-t}/N$. If a t-subset regular k-hypergraph X of order n is self-complementary, then X and its complement X^C are both t-(n, k, λ) designs with $\lambda = \binom{n-t}{k-t}/2$. Hence the pair $\{X, X^C\}$ is a $LS[2](t, k, n)$ in which the t-designs are isomorphic. If X is t-fold-transitive, then the corresponding t-design is also t-fold-transitive. Hence vertex transitive self-complementary k-hypergraphs of order n correspond bijectively to large sets of t-designs $LS[2](t, k, n)$ with $t \geq 1$ in which the t-designs are point-transitive and isomorphic.

In this thesis, however, we will use terminology from hypergraph theory, rather than design theory.

Let p^r be the highest power of a prime p dividing the order of a finite group G. A *Sylow p-subgroup* of G is a subgroup of G of order p^r. We will make use of the following notation for permutation groups. For a finite set Ω, a point v in Ω, a permutation τ on Ω, a permutation group G on Ω, and a prime p, the symbols $v^\tau, v^G, \tau^{-1}G\tau$, and $Syl_p(G)$ will denote the image of v by τ, the orbit of G containing v, the conjugate of G by τ, and the set of all Sylow p-subgroups of G, respectively. The *stabilizer* of the point v in the group G is denoted by G_v and defined by $G_v = \{\tau \in G : v^\tau = v\}$. For a subset $\Delta \subseteq \Omega$, the *set-stabilizer* of the set Δ in G is denoted by G_Δ and defined by $G_\Delta = \{\tau \in G : \Delta^\tau = \Delta\}$, where $\Delta^\tau = \{v^\tau : v \in \Delta\}$. It is not difficult to show that the stabilizer G_v and the set-stabilizer G_Δ are each subgroups of G. For finite sets Ω and Π, a permutation α of Ω, and a permutation β of Π, the permutation $\alpha \times \beta$ of $\Omega \times \Pi$ is defined by

$$(u, v)^{\alpha\times\beta} = (u^\alpha, v^\beta), \quad \text{for all } (u, v) \in \Omega \times \Pi.$$

For groups $G \leq Sym(\Omega)$ and $H \leq Sym(\Pi)$, the symbol $G \times H$ denotes the set of permutations $\{\alpha \times \beta : \alpha \in G, \beta \in H\}$. One can easily verify that $G \times H$ is a subgroup of $Sym(\Omega \times \Pi)$.

For positive integers m and n, let $n_{[m]}$ denote the unique integer in $\{0, 1, \ldots, m-1\}$ such that $n \equiv n_{[m]} (\bmod\ m)$. Thus $n_{[m]}$ is the remainder upon division of n by

m. Let $\left[\frac{n}{m}\right]$ denote the quotient upon division of n by m. Finally, for any prime number p, let $n_{(p)}$ denote the largest integer i such that p^i divides n. We will denote the *binary representation* of an integer k by a vector $b = (b_m, b_{m-1}, \ldots, b_1, b_0)_2$. This is, the entries of the vector b satisfy $k = \sum_{i=0}^{m} b_i 2^i$, $b_m = 1$, and $b_i \in \{0, 1\}$ for all $i \in \{0, 1, \ldots, m\}$. The *support* of the binary representation b is the set $\{i \in \{0, 1, 2, \ldots, m\} : b_i = 1\}$, and is denoted by $\text{supp}(b)$.

1.2 History and layout of part I

Much of the research to date into self-complementary uniform hypergraphs has been focused on determining necessary and sufficient conditions on the order of these structures. In the early 1960s, Sachs [29] and Ringel [28] determined necessary and sufficient conditions on the order n of a self-complementary graph (2-hypergraph). They used a simple counting argument to show that n must be congruent to 0 or 1 modulo 4, and then they characterized the lengths of the cycles in the disjoint cycle decomposition of any graph antimorphism, giving an algorithm for generating all self-complementary graphs of a given order n. In particular, they showed that there exists a self-complementary graph of every admissible order $n \equiv 0, 1 (\text{mod } 4)$. In 1978, M.J. Colbourn and C.J. Colbourn [7] showed that one of the most important problems in graph theory, the graph isomorphism problem, is polynomially equivalent to the problem of determining whether two self-complementary graphs are isomorphic. Since then, there has been a great deal of research into self-complementary graphs. A good reference on self-complementary graphs and their generalizations was written by A. Farrugia [10].

In Part I of this thesis, we focus on the generalization of self-complementary graphs to self-complementary k-uniform hypergraphs. Part I is divided into four chapters.

Generating hypergraphs

In Chapter 2, we present necessary and sufficient conditions on the order n of a self-complementary k-uniform hypergraph, and we discuss a method for generating all of these structures up to isomorphism, for feasible n.

In 1985, Suprunenko [30] generalized the method by Ringel and Sachs for generating all self-complementary graphs to find a method for generating all self-complementary 3-hypergraphs. His characterization of the cycle type of an antimorphism of a 3-hypergraph was also found independently by Kocay [19] in 1992. In 2005, Szymański took this method one step further to characterize the cycle type of an antimorphism of a 4-hypergraph, and gave an algorithm to generate all self-complementary 4-hypergraphs of a given order n. In 2007, Wojda gave a general characterization of the cycle type of an antimorphism of a k-hypergraph. Wojda's characterization is stated in Theorem 2.2.4. However, given a permutation in $Sym(n)$, it is difficult to determine whether Wojda's condition holds. In Theorem 2.2.5, we give a more transparent characterization of the cycle type of a k-complementing permutation in $Sym(n)$ which has order equal to a power of 2. This yields a test to determine whether or not a finite permutation is a k-complementing permutation (see Corollary 2.2.7 and Algorithm 2.4.4), and an algorithm for generating all self-complementary k-hypergraphs of order n, up to isomorphism, for all feasible n and k (see Algorithm 2.4.3). This extends the previous results for the cases $k = 2, 3, 4$ due to Ringel, Sachs, Suprunenko, Kocay and Szymański.

In 2007, Szymański and Wojda proved that for positive integers n and k with $k \leq n$, a self-complementary k-uniform hypergraph of order n exists if and only if $\binom{n}{k}$ is even. Our characterization of the cycle type of a k-complementing permutation in $Sym(n)$ gives an alternative description of this necessary and sufficient condition on the order of a self-complementary k-uniform hypergraph in terms of the binary representation of k (see Corollary 2.3.2). This yields more transparent conditions on order in the case where k is a sum of consecutive powers of 2 (see Corollary 2.3.4).

Regular hypergraphs

In Chapter 3, we examine the orders of t-subset-regular self-complementary k-uniform hypergraphs.

In 1975, Hartman showed that a necessary condition for the existence of a $LS[2](t,k,n)$ is that $\binom{n-i}{k-i}$ is even for all $i \in \{0,1,\ldots,t\}$, and conjectured that these necessary conditions are also sufficient. In 2003, Khosrovshahi and Tayfeh-Rezaie[17] gave a useful and equivalent description of these necessary conditions as a set of congruence relations. This gives necessary conditions on the order of a t-subset-regular self-complementary k-hypergraph.

In 1998, Ajoodani-Namini proved that Hartman's conjecture was true for $t = 1, 2$. However, this does not prove that there exist 1- and 2-subset-regular self-complementary k-hypergraphs for every n satisfying Hartman's necessary conditions, since there is no guarantee that the t-designs in a $LS[2](t,k,n)$ are isomorphic. In 1985, Rao [27] constructed regular self-complementary graphs (2-hypergraphs) of every admissible order n (that is, $n \equiv 1 \pmod 4$). In 2007, Potočnik and Šajna [23] found constructions for 1-subset-regular self-complementary 3-hypergraphs of every admissible order n (that is, $n \equiv 1, 2 \pmod 4$). In 2008, Knor and Potočnik [18] constructed 2-subset-regular self-complementary 3-hypergraphs of every admissible order n (that is, $n \equiv 2 \pmod 4$). Hence Hartman's necessary conditions are sufficient in the cases where $k \in \{2, 3\}$. We will state these previous results in Chapter 3.

In Theorem 2.3.5, we reformulate Khosrovshahi and Tayfeh-Rezaie's necessary conditions on the order of a t-subset-regular self-complementary k-hypergraph in terms of the binary representation of k. This yields more transparent necessary conditions on the order of these structures in the case where the rank k is a sum of consecutive powers of 2. In addition, we prove that Khosrovshahi and Tayfeh-Rezaie's necessary conditions are in fact sufficient for all k in the case $t = 1$. This yields the main result of this section, Theorem 3.2.6, which states that a 1-subset-regular self-complementary k-hypergraph of order n exists if and only if $1 \leq n_{[a]} < k_{[2^a]}$ for some integer a such that $\max\{i : 2^i \mid k\} < a \leq \min\{i : 2^i > k\}$. We conclude the chapter with some open problems, and we discuss their connection to design theory.

Transitive hypergraphs

In Chapter 4, we examine the orders of t-fold transitive self-complementary k-uniform hypergraphs.

In 1985, Rao [27] constructed vertex transitive self-complementary graphs of all orders n for which the highest power p^r of any prime p dividing n satisfies $p^r \equiv 1 \pmod 4$, and conjectured that these sufficient conditions on the order of a transitive self-complementary graph may also be necessary. In 1997, Li [20] proved Rao's conjecture was correct in the case when n is a product of distinct primes. In 1999, Muzychuk [21] gave a group-theoretic proof of Rao's conjecture for all n. Hence a vertex transitive self-complementary graph of order n exists if and only if the highest power p^r of any prime p dividing n is congruent to 1 modulo 4. In 2007, Potočnik and Šajna[24] generalized Muzychuk's result and showed that, if $k = 2^\ell$ or $k = 2^\ell + 1$ and there exists a vertex transitive k-hypergraph of order $n \equiv 1 \pmod{2^{\ell+1}}$, then the highest power p^r of any prime p dividing n must be congruent to 1 modulo $2^{\ell+1}$. In Section 4.1.1, we state the results to date regarding necessary conditions on the order of vertex transitive self-complementary uniform hypergraphs. In Section 4.1.2, Theorem 4.1.3, we extend Potočnik and Šajna's result inductively to give necessary conditions on the order of a t-fold transitive k-hypergraph of order $n \equiv t \pmod{2^{\ell+1}}$, for all $t \in \{1, 2, \ldots, k-1\}$.

Potočnik and Šajna also gave constructions of vertex transitive self-complementary 3-hypergraphs, and showed that if $k = 3$, their necessary condition is also sufficient, consequently generalizing Rao and Muzychuk's result to 3-hypergraphs of odd order. They also constructed many other vertex transitive self-complementary k-hypergraphs, and obtained more sufficient conditions on the order of these structures. We state these sufficient conditions in Section 4.2.1. In Section 4.2.2, we present Construction 4.2.8 for vertex transitive self-complementary 2^ℓ- and $(2^\ell + 1)$-hypergraphs of any order n satisfying Potočnik and Šajna's necessary conditions, and consequently we prove that their necessary conditions are sufficient for these ranks. This yields Theorem 4.2.10, the main result of this chapter. We close Chapter 4 with some open problems.

Transitive hypergraphs of prime order

In Chapter 5, we use a characterization of the transitive groups of prime degree due to Burnside [35] and Zassenhaus [38] to determine the group of automorphisms and antimorphisms of the vertex transitive self-complementary k-hypergraphs of prime order $p \equiv 1 \pmod{2^{\ell+1}}$ in the case where $k = 2^\ell$ or $k = 2^\ell + 1$. We use this information to generate all such hypergraphs in Algorithm 5.3.1. As a consequence, we obtain a bound on the number of pair-wise non-isomorphic vertex transitive self-complementary graphs of prime order $p \equiv 1 \pmod 4$ (see Corollary 5.3.4). We conclude Chapter 5 with some open problems.

Chapter 2

Generating self-complementary hypergraphs

In Section 2.1, we discuss a method for generating self-complementary k-hypergraphs of order n given a k-complementing permutation in $Sym(n)$, which yields a simple characterization of k-complementing permutations and a method for generating all of the self-complementary k-hypergraphs having a given antimorphism θ.

In Section 2.2 we present some results due to Ringel, Sachs, Suprunenko, Kocay, and Szymański regarding the cycle type of an antimorphism of a self-complementary k-hypergraph for $k = 2, 3, 4$. We also present a general characterization of the cycle type of a k-complementing permutation in $Sym(n)$ due to Wojda. Then we present a new result, Theorem 2.2.5, which characterizes the cycle type of a k-complementing permutation whose order is a power of 2, for any positive integer k. This yields a test to determine whether or not a finite permutation is a k-complementing permutation. It also yields an algorithm for generating all of the self-complementary k-hypergraphs of order n, up to isomorphism, which will be presented in Section 2.4.

In Section 2.3, we state Szymański and Wojda's necessary and sufficient condition on the order n of a self-complementary k-hypergraph, namely that $\binom{n}{k}$ is even. Then we use Theorem 2.2.5 to obtain an alternative description of this condition in terms of the binary representation of k (see Corollary 2.3.2). This yields more transparent conditions on the order of a self-complementary k-hypergraph when the rank k is a sum of consecutive powers of 2.

2.1 Complementing permutations

In this section, we obtain a simple characterization of the permutations in $Sym(n)$ which are antimorphisms of self-complementary k-hypergraphs, and we present an algorithm which generates all of the self-complementary k-hypergraphs that have a given antimorphism $\theta \in Sym(n)$.

In some of the literature ([19], [31] and [39]), a permutation which is an antimorphism of a self-complementary k-hypergraph is called a *k-complementing permutation*. In [31] and [39], Szymański and Zwonek demonstrate a close correspondence between the class of all self-complementary k-hypergraphs of order n and the set of k-complementing permutations on $\{1, 2, \ldots, n\}$. Let θ be any permutation on $V = \{1, 2, \ldots, n\}$. Then one may try to construct a self-complementary k-hypergraph X induced by θ as follows: Take any $A_1 \in V^{(k)}$, and define X to be the k-hypergraph in which

$$A_1^{\theta^j} \in E(X) \quad \Longleftrightarrow \quad j \text{ is even.} \tag{2.1.1}$$

Now take $A_2 \in V^{(k)} \setminus A_1^{\langle\theta\rangle}$, and define more elements of X as in (2.1.1), but this time with A_1 replaced by A_2. Proceed in this way until we have exhausted all elements of $V^{(k)}$. This procedure leads to a well-defined self-complementary hypergraph if and only if

$$A^{\theta^j} \neq A \text{ for all } A \in V^{(k)} \text{ and for all } j \text{ odd.} \tag{2.1.2}$$

Note that condition (2.1.2) holds if an only if the sequence

$$A, \quad A^{\theta}, \quad A^{\theta^2}, \quad A^{\theta^3}, \quad \ldots$$

has even length. We obtain the following result.

Proposition 2.1.1 [36] *Let V be a finite set, let k be a positive integer such that $k \leq |V|$, and let $\theta \in Sym(V)$. Then the following three statements are equivalent:*

1. *θ is a k-complementing permutation.*

2. *$A^{\theta^j} \neq A$ for all $A \in V^{(k)}$ and for all j odd.*

3. *The sequence $A, A^{\theta}, A^{\theta^2}, A^{\theta^3}, \ldots$ has even length for all $A \in V^{(k)}$.*

The method described above yields the following algorithm, which takes a k-complementing permutation in $Sym(V)$ as input, and returns the set $\mathcal{H}_\theta$ of all self-complementary k-hypergraphs X on V that have θ as an antimorphism. This algorithm was previously described by Sachs [29] and Ringel [28] for $k = 2$, by Suprunenko [30] for $k = 2, 3$, and by Szymański [31] for $k = 3, 4$.

Algorithm 2.1.2 [29],[28], [30],[31]
Let V be a finite set, let k be a positive integer such that $k \leq |V|$, and let θ be a k-complementing permutation in $Sym(V)$.

(1) Set $\mathcal{H}_\theta := \emptyset$.

(a) In steps (a)(i) and (a)(ii) we will find each orbit of θ on $V^{(k)}$ and colour each element of $V^{(k)}$ either red or blue.

(i) Take an arbitrary uncoloured element $A \in V^{(k)}$, and create a sequence

$$A, \quad A^{\theta}, \quad A^{\theta^2}, \quad A^{\theta^3}, \quad \ldots$$

This sequence is an orbit of θ on $V^{(k)}$, and its length is a divisor of $|\theta|$. Colour the edges of the form $A^{\theta^{2i}}$ red and those of the form $A^{\theta^{2i+1}}$ blue. Since θ is a k-complementing permutation, Proposition 2.1.1 guarantees that there are no edges of $V^{(k)}$ coloured both red and blue.

(ii) Repeat step (a)(i) for any uncoloured edges of $V^{(k)}$, until all edges have been coloured.

(b) Let m be the number of orbits of θ on $V^{(k)}$ constructed in step (a), and choose an ordering $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_m$ of these orbits. Set $W = \mathbb{Z}_2^m$.

(i) Choose a vector $w \in W$.

Let $X(\theta, w)$ be the k-hypergraph with vertex set V and edge set E, where an edge $e \in \mathcal{O}_i$ is in E if and only if e is red and $w_i = 1$, or e is blue and $w_i = 0$. Then $X(\theta, w)$ is a self-complementary k-uniform hypergraph.

Set $\mathcal{H}_\theta := \mathcal{H}_\theta \cup \{X(\theta, w)\}$. Set $W := W \setminus \{w, \mathbf{1} - w\}$.

(ii) Repeat step (b)(i) until $W = \emptyset$.

(2) Return $\mathcal{H}_\theta$.

Note that $X(\theta, w)$ is isomorphic to its complement $X(\theta, \mathbf{1} - w)$, where $\mathbf{1}$ is the vector in $\mathbb{Z}_2^m$ with every entry equal to 1. Thus, for each k-complementing permutation θ in $Sym(V)$, Algorithm 2.1.2 will generate the set $\mathcal{H}_\theta$ of all self-complementary k-hypergraphs on V for which θ is an antimorphism, up to isomorphism. That is, every self-complementary k-hypergraph on V for which θ is an antimorphism is isomorphic to one of the hypergraphs in $\mathcal{H}_\theta$. The set $\mathcal{H}_\theta$ is called the *θ-switching class* of self-complementary k-hypergraphs on V. Any two self-complementary k-hypergraphs in this θ-switching class are said to be *θ-switching equivalent*, and each self-complementary k-hypergraph in this θ-switching class is said to be *induced by θ*.

2.2 Cycle types of antimorphisms

In this section, we will characterize the cycle type of a k-complementing permutation in $Sym(n)$ whose order is equal to a power of 2. Whenever we refer to a cycle of a permutation θ, we mean a cycle in the disjoint cycle decomposition of θ.

2.2.1 Previous results

The following is a well-known result regarding the cycle types of antimorphisms of self-complementary graphs (2-hypergraphs). It was originally proved by Sachs [29] and Ringel [28], but a proof can also be found in Suprunenko [30].

Lemma 2.2.1 [29, 28, 30] *A permutation θ is an antimorphism for a self-complementary graph if and only if one of the following hold:*

(i) *The length of every cycle of θ is divisible by 4.*

(ii) *θ has exactly one fixed point, and all other cycles have length divisible by 4.*

Suprunenko [30] proved the following analogue to Lemma 2.2.1 for 3-hypergraphs. This result was also proved later by Kocay in [19].

Lemma 2.2.2 [30, 19] *A permutation θ is an antimorphism of a self-complementary 3-hypergraph if and only if one of the following hold:*

(i) *Every cycle of θ has even length.*

(ii) *θ has one or two fixed points, and all other cycles have length divisible by 4.*

Szymański [31] took this method a step further, and proved the following analogue of Lemmas 2.2.1 and 2.2.2 for 4-hypergraphs.

Lemma 2.2.3 [31] *A permutation θ is an antimorphism for a self-complementary 4-hypergraph if and only if one of the following hold:*

(i) *The length of every cycle of θ is divisible by 8.*

(ii) *θ has one, two or three fixed points and all other cycles have length divisible by 8.*

(iii) *θ has one cycle of length 2, and all other cycles have length divisible by 8.*

(iv) *θ has one fixed point, 1 cycle of length 2, and all other cycles have length divisible by 8.*

(v) *θ has one cycle of length 3, all other cycles have length divisible by 8.*

The next result is due to Wojda [36, 33], and it gives necessary and sufficient conditions on the cardinality of the orbits of a k-complementing permutation.

Theorem 2.2.4 **[36, 33]** *Let k and m be positive integers, let V be a finite set, and let $\sigma \in Sym(V)$ with orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, O_m$. Let $2^{q_i}(2s_i+1)$ denote the cardinality of the orbit $\mathcal{O}_i$, for $i = 1, 2, \ldots, m$. The permutation σ is a k-complementing permutation if and only if, for every $\ell \in \{1, 2, \ldots, k\}$ and for every decomposition*

$$k = k_1 + k_2 + \cdots + k_\ell$$

of k, where $k_j = 2^{p_j}(2r_j + 1)$ for nonnegative integers p_j and r_j, and for every subsequence of orbits

$$\mathcal{O}_{i_1}, \mathcal{O}_{i_2}, \ldots, \mathcal{O}_{i_\ell}$$

such that $k_j \leq |\mathcal{O}_{i_j}|$ for $j = 1, 2, \ldots, \ell$, there is a subscript $j_0 \in \{1, 2, \ldots, \ell\}$ such that $p_{j_0} < q_{i_{j_0}}$.

Given a permutation, it is difficult to determine whether Wojda's condition holds. In Theorem 2.2.5 we give a more transparent characterization of the orbit lengths of k-complementing permutations which have order equal to a power of 2, and Corollary 2.2.7 and Algorithm 2.4.4 will show how we can use our characterization to test whether a finite permutation is a k-complementing permutation. In Section 2.4, we will use the characterization of Theorem 2.2.5 to obtain Algorithm 2.4.3 for generating all of the self-complementary k-hypergraphs of order n, up to isomorphism, for feasible k and n.

2.2.2 New results

Theorem 2.2.5 below gives a characterization of the cycle types of k-complementing permutations which have order equal to a power of 2 in terms of the binary representation of k. We will show that this is sufficient to characterize all of the k-complementing permutations, and that it leads to an algorithm for generating all of the self-complementary k-hypergraphs of order n for feasible n and k.

Recall the definition of the binary representation of k, and the notation $\text{supp}(b)$ and $n_{[m]}$ from page 5 of Section 1.1.

Theorem 2.2.5 *Let V be a finite set, let k be a positive integer such that $k \leq |V|$, and let $b = (b_m, b_{m-1}, \ldots, b_2, b_1, b_0)_2$ be the binary representation of k. Let $\theta \in Sym(V)$ be a permutation whose order is a power of 2. Given $\ell \in supp(b)$, let A_ℓ denote the set of those points of V contained in cycles of θ of length $< 2^\ell$, and let B_ℓ denote the set of those points of V contained in cycles of θ of length $> 2^\ell$. Then θ is a k-complementing permutation if and only if, for some $\ell \in supp(b)$, $V = A_\ell \cup B_\ell$ and $|A_\ell| < k_{[2^{\ell+1}]}$.*

Proof: ($\Rightarrow$) Suppose that θ is a k-complementing permutation of order a power of 2. Then every cycle of θ has length a power of 2. If θ contained a cycle of length 2^i for every $i \in \text{supp}(b)$, then there would be an invariant set of θ of cardinality $\sum_{i \in \text{supp}(b)} 2^i = k$, a contradiction. Hence, for some $\ell \in \text{supp}(b)$, θ does not contain a cycle of length 2^ℓ.

Let

$$L = \{\ell \in \text{supp}(b) : \theta \text{ does not contain a cycle of length } 2^\ell\}. \tag{2.2.1}$$

Then $V = A_\ell \cup B_\ell$ for all $\ell \in L$. It remains to show that $|A_\ell| < k_{[2^{\ell+1}]}$ for some $\ell \in L$.

Suppose to the contrary that $|A_\ell| \geq k_{[2^{\ell+1}]}$ for all $\ell \in L$. Write $|A_\ell| = \sum_{i=0}^{\ell-1} a_i 2^i$, where a_i is the number of cycles of θ of length 2^i. Note that $k_{[2^{\ell+1}]} = \sum_{i=0}^{\ell} b_i 2^i$. Thus, by assumption, $|A_\ell| \geq \sum_{i=0}^{\ell} b_i 2^i$ for all $\ell \in L$. Suppose $L = \{\ell_1, \ell_2, \ldots, \ell_t\}$ where $\ell_1 < \ell_2 < \cdots < \ell_t$.

- **Claim A:** Let $x \in \{1, 2, \ldots, t\}$. If $|A_{\ell_j}| \geq \sum_{i=0}^{\ell_j} b_i 2^i$ for all $j \in \{1, 2, \ldots, x\}$, then $\theta|_{A_{\ell_x}}$ has an invariant set of size $\sum_{i=0}^{\ell_x} b_i 2^i$.

Proof of Claim A: The proof is by induction on x. First we will need some notation. For any nonnegative integer i, let a_i denote the number of cycles of θ of length 2^i. Then certainly $\sum_{i=0}^{\ell_j - 1} a_i 2^i = |A_{\ell_j}|$, for $j = 1, 2, \ldots, t$. Also, for any sequence of integers $\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{\ell_j - 1}$ such that $0 \leq \hat{a}_i \leq a_i$ for $0 \leq i \leq \ell_j - 1$, the sum $\sum_{i=0}^{\ell_j - 1} \hat{a}_i 2^i$ is the sum of the lengths of a collection of cycles of $\theta|_{A_{\ell_j}}$, and hence it is the size of an invariant set of $\theta|_{A_{\ell_j}}$. Conversely, any invariant set S of $\theta|_{A_{\ell_j}}$ corresponds to a collection of cycles of $\theta|_{A_{\ell_j}}$ whose lengths sum to $|S|$, and hence there exists a sequence of integers $\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{\ell_j - 1}$ such that $0 \leq \hat{a}_i \leq a_i$ for $0 \leq i \leq \ell_j - 1$, and $|S| = \sum_{i=0}^{\ell_j - 1} \hat{a}_i 2^i$.

Base Step: If $x = 1$ and $|A_{\ell_1}| \geq \sum_{i=0}^{\ell_1} b_i 2^i$, then

$$|A_{\ell_1}| = \sum_{i=0}^{\ell_1 - 1} a_i 2^i \geq \sum_{i=0}^{\ell_1} b_i 2^i. \tag{2.2.2}$$

By the definition of L in (2.2.1), it follows that $a_i \geq b_i$ for $0 \leq i \leq \ell_1 - 1$. Hence (2.2.2) implies that

$$\sum_{i=0}^{\ell_1 - 1} (a_i - b_i) 2^i \geq 2^{\ell_1}$$

holds with $a_i - b_i \geq 0$ for all $i = 1, 2, \ldots, \ell_1 - 1$. Thus by Lemma A.0.14 (see Appendix), there is a sequence $c_0, c_1, \ldots, c_{\ell_1 - 1}$ such that $0 \leq c_i \leq (a_i - b_i)$ for $0 \leq i \leq \ell_1 - 1$, and

$$\sum_{i=0}^{\ell_1 - 1} c_i 2^i = 2^{\ell_1}.$$

Now let $\hat{a}_i = b_i + c_i$. Then

$$0 \leq \hat{a}_i = b_i + c_i \leq b_i + (a_i - b_i) = a_i$$

and hence

$$0 \leq \hat{a}_i \leq a_i$$

for $0 \leq i \leq \ell_1 - 1$, and

$$\sum_{i=0}^{\ell_1-1} \hat{a}_i 2^i = \sum_{i=0}^{\ell_1-1} b_i 2^i + \sum_{i=0}^{\ell_1-1} c_i 2^i = \sum_{i=0}^{\ell_1-1} b_i 2^i + 2^{\ell_1} = \sum_{i=0}^{\ell_1} b_i 2^i.$$

Thus $\theta|_{A_{\ell_1}}$ has an invariant set of size $\sum_{i=0}^{\ell_1} b_i 2^i$, as required.

Induction Step: Let $2 \leq x \leq t$ and assume that if $|A_{\ell_j}| \geq \sum_{i=0}^{\ell_j} b_i 2^i$ for all $j \in \{1, 2, \ldots, x-1\}$, then $\theta|_{A_{\ell_{x-1}}}$ has an invariant set of size $\sum_{i=0}^{\ell_{x-1}} b_i 2^i$. Now suppose that $|A_{\ell_j}| \geq \sum_{i=0}^{\ell_j} b_i 2^i$ for all $j \in \{1, 2, \ldots, x\}$. Then certainly $|A_{\ell_j}| \geq \sum_{i=0}^{\ell_j} b_i 2^i$ for all $j \in \{1, 2, \ldots, x-1\}$, and so by the induction hypothesis, $\theta|_{A_{\ell_{x-1}}}$ has an invariant set of size $\sum_{i=0}^{\ell_{x-1}} b_i 2^i$. This implies that there is a sequence of integers $c_0, c_1, \ldots, c_{\ell_{x-1}-1}$ such that $0 \leq c_i \leq a_i$ for $0 \leq i \leq \ell_{x-1} - 1$, and

$$\sum_{i=0}^{\ell_{x-1}-1} c_i 2^i = \sum_{i=0}^{\ell_{x-1}} b_i 2^i. \tag{2.2.3}$$

Since $|A_{\ell_x}| \geq \sum_{i=0}^{\ell_x} b_i 2^j$, we have

$$\sum_{i=0}^{\ell_x-1} a_i 2^i \geq \sum_{i=0}^{\ell_x} b_i 2^i. \tag{2.2.4}$$

Since $\ell_{x-1} \in L$, $a_{\ell_{x-1}} = 0$, so (2.2.4) implies that

$$|A_{\ell_x}| = \sum_{i=0}^{\ell_x-1} a_i 2^i = \sum_{i=0}^{\ell_{x-1}-1} a_i 2^i + \sum_{i=\ell_{x-1}+1}^{\ell_x-1} a_i 2^i \geq \sum_{i=0}^{\ell_x} b_i 2^i.$$

Hence by (2.2.3), we have

$$\sum_{i=0}^{\ell_{x-1}-1} (a_i - c_i) 2^i + \sum_{i=\ell_{x-1}+1}^{\ell_x-1} a_i 2^i \geq \sum_{i=\ell_{x-1}+1}^{\ell_x} b_i 2^i.$$

This implies that

$$\sum_{i=0}^{\ell_{x-1}-1}(a_i-c_i)2^i+\sum_{i=\ell_{x-1}+1}^{\ell_x-1}(a_i-b_i)2^i \geq 2^{\ell_x}. \tag{2.2.5}$$

By the definition of L in (2.2.1), we have $a_i - b_i \geq 0$ for $\ell_{x-1}+1 \leq i \leq \ell_x - 1$. Also, $a_i - c_i \geq 0$ for $0 \leq i \leq \ell_{x-1}-1$. Thus (2.2.5) and Lemma A.0.14 with $n = 2$ (see Appendix) guarantee that there exists a sequence of integers $d_0, d_1, \ldots, d_{\ell_x-1}$ such that $0 \leq d_i \leq a_i - c_i$ for all $i \in \{0, 1, \ldots, \ell_{x-1}-1\}$, $d_{\ell_{x-1}} = 0$, $0 \leq d_i \leq a_i - b_i$ for all integers i such that $\ell_{x-1}+1 \leq i \leq \ell_x - 1$, and

$$\sum_{i=0}^{\ell_x-1} d_i 2^i = 2^{\ell_x}. \tag{2.2.6}$$

Now define a sequence of integers $\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{\ell_x-1}$ by

$$\hat{a}_i = \begin{cases} c_i + d_i, & \text{if } 0 \leq i \leq \ell_{x-1}-1 \\ 0, & \text{if } i = \ell_{x-1} \\ b_i + d_i, & \text{if } \ell_{x-1}+1 \leq i \leq \ell_x - 1 \end{cases}.$$

Now for $i = 0, 1, \ldots, \ell_{x-1}-1$, we have $0 \leq d_i \leq a_i - c_1$, and thus $0 \leq c_i + d_i \leq c_i + (a_i - c_i) = a_i$. Hence $0 \leq \hat{a}_i \leq a_i$ for these i. Moreover, for each integer i such that $\ell_{x-1}+1 \leq i \leq \ell_x - 1$, we have $0 \leq d_i \leq a_i - b_i$, and thus $0 \leq b_i + d_i \leq b_i + (a_i - b_1) = a_i$. Hence $0 \leq \hat{a}_i \leq a_i$ for these i also. Since $\hat{a}_{\ell_{x-1}} = 0$, we conclude that $0 \leq \hat{a}_i \leq a_i$ for $i = 0, 1, \ldots, \ell_x - 1$. Moreover,

$$\begin{aligned}
\sum_{i=0}^{\ell_x-1}\hat{a}_i 2^i &= \sum_{i=0}^{\ell_{x-1}-1}(c_i+d_i) + 0 + \sum_{i=\ell_{x-1}+1}^{\ell_x-1}(b_i+d_i) \\
&= \sum_{i=0}^{\ell_{x-1}-1} c_i + \sum_{i=\ell_{x-1}+1}^{\ell_x-1} b_i + \sum_{i=0}^{\ell_x-1} d_i \quad (\text{since } d_{\ell_{x-1}} = 0) \\
&= \sum_{i=0}^{\ell_{x-1}} b_i + \sum_{i=\ell_{x-1}+1}^{\ell_x-1} b_i + \sum_{i=0}^{\ell_x-1} d_i \quad (\text{by (2.2.3)})
\end{aligned}$$

$$= \sum_{i=0}^{\ell_x - 1} b_i 2^i + 2^{\ell_x} \quad \text{(by (2.2.6))}$$
$$= \sum_{i=0}^{\ell_x} b_i 2^i. \quad \text{(since } b_{\ell_x} = 1\text{)}$$

Thus $\theta|_{A_{\ell_x}}$ has an invariant set of size $\sum_{i=0}^{\ell_x} b_i 2^i$, as required.

Hence by mathematical induction, Claim A holds for all $x \in \{1, 2, \ldots, t\}$.

Now applying Claim A with $x = t$, we observe that $|A_{\ell_j}| \geq \sum_{i=0}^{\ell_j} b_i 2^i$ for all $j \in \{1, 2, \ldots, t\}$. Hence $\theta|_{A_{\ell_t}}$ has an invariant set of size $\sum_{i=0}^{\ell_t} b_i 2^i$. But since ℓ_t is the largest element of L, $\theta|_{B_{\ell_t}}$ (and hence θ) contains a cycle of length 2^ℓ for all $\ell \in \text{supp}(b)$ with $\ell_t < \ell \leq m$, and hence θ contains an invariant set of size $\sum_{i=0}^{m} b_i 2^i = k$. This contradicts the fact that θ is a k-complementing permutation.

We conclude that for some $j \in \{1, 2, \ldots, t\}$, $|A_{\ell_j}| < \sum_{i=0}^{\ell_j} b_i 2^i$. For this j, set $\ell = \ell_j$. Then $\ell \in \text{supp}(b)$ and $|A_\ell| < k_{[2^{\ell+1}]}$, as required.

($\Leftarrow$) Let $\theta \in Sym(V)$ with order a power of 2 and suppose that, for some $\ell \in \text{supp}(b)$, $V = A_\ell \cup B_\ell$ and $|A_\ell| < k_{[2^{\ell+1}]}$. This implies that θ does not have an invariant set of size k. Moreover, since the order of θ is a power of 2, for each odd integer j, θ^j has the same cycle type as θ, and hence θ^j also has no invariant set of size k. Hence $A^{\theta^j} \neq A$ for all odd integers j and all $A \in V^{(k)}$, and so Proposition 2.1.1 implies that θ is a k-complementing permutation. ∎

Theorem 2.2.5 together with the following lemma yields Corollary 2.2.7, which gives a test to determine whether a given permutation in $Sym(n)$ is a k-complementing permutation.

Lemma 2.2.6 *Let k be a positive integer.*

(1) Let s be a nonnegative integer. A permutation θ is a k-complementing permutation if and only if θ^{2s+1} is a k-complementing permutation.

(2) A self-complementary k-hypergraph has an antimorphism whose order is equal to a power of 2.

Proof:

1. If $\theta \in Sym(V)$ is a k-complementing permutation, then $\theta \in \mathrm{Ant}(X)$ for some self-complementary k-hypergraph $X = (V, E)$, and so θ is a bijection from E to E^C and a bijection from E^C to E. It follows that $\theta^{2s+1} \in \mathrm{Ant}(X)$.

   Conversely, suppose that θ^{2s+1} is a k-complementing permutation. Then Proposition 2.1.1 guarantees that each orbit of θ^{2s+1} on $V^{(k)}$ has even cardinality. Observe that each orbit of θ^{2s+1} on $V^{(k)}$ is contained in an orbit of θ on $V^{(k)}$. Also, every k-subset in an orbit of θ on $V^{(k)}$ must certainly lie in an orbit of θ^{2s+1} on $V^{(k)}$. Since the orbits of θ^{2s+1} on $V^{(k)}$ are pairwise disjoint, it follows that every orbit of θ on $V^{(k)}$ is a union of pairwise disjoint orbits of θ^{2s+1} on $V^{(k)}$, each of which has even cardinality. Hence every orbit of θ on $V^{(k)}$ has even cardinality, and so Proposition 2.1.1 implies that θ is a k-complementing permutation.

2. Let X be a self-complementary k-hypergraph, and let $\theta \in \mathrm{Ant}(X)$. Proposition 2.1.1 guarantees that θ has even order, so $|\theta| = 2^z s$ for some positive integer z and some odd integer s. Since s is odd, part (1) implies that $\theta^s \in \mathrm{Ant}(X)$, and θ^s has order equal to a power of 2.

∎

Thus Lemma 2.2.6(1) and Theorem 2.2.5 together yield the following characterization of k-complementing permutations.

Corollary 2.2.7 *Let k be a positive integer, let b be the binary representation of k, and let V be a finite set. A permutation $\sigma \in Sym(V)$ is a k-complementing permutation if and only if $|\sigma| = 2^i(2t+1)$ for some integers t and i such that $i \geq 1$ and $t \geq 0$, and $\theta = \sigma^{2t+1}$ satisfies the conditions of Theorem 2.2.5 for some $\ell \in supp(b)$.* ∎

In Algorithm 2.4.4, we will use the conditions of Corollary 2.2.7 to derive a straightforward method to test whether a given $\theta \in Sym(V)$ is a k-complementing permutation.

The following corollary to Theorem 2.2.5 was first proved directly by Potočnik and Šajna [24]. We will need to make use of this result in Chapter 5, in the proof of Theorem 5.3.3.

Corollary 2.2.8 [24] *Let ℓ be a positive integer, let $k = 2^\ell$ or $k = 2^\ell + 1$, and let $n \equiv 1(\text{mod } 2^{\ell+1})$. Let X be a self-complementary k-hypergraph of order n. Let $\mathcal{O}_2$ be the set of elements of $Ant(X)$ whose orders are powers of 2. Then every element of $\mathcal{O}_2$ has exactly one fixed point and all other orbits have length divisible by $2^{\ell+1}$.*

Proof: Let $\theta \in \mathcal{O}_2$ and let b be the binary representation of k. Then $\text{supp}(b) \subseteq \{0, \ell\}$. Now the conditions of Theorem 2.2.5 must hold for θ for some $\hat{\ell} \in \text{supp}(b)$. If these conditions hold with $\hat{\ell} = 0$, then each cycle of θ has length equal to 2^r for $r > 0$. But this implies that $n =\equiv 0$ (mod 2), contradicting the fact that $n \equiv 1$ (mod $2^{\ell+1}$). Hence the conditions of Theorem 2.2.5 must hold for ℓ. Thus $V = A_\ell \cup B_\ell$ and $|A_\ell| < k_{[2^{\ell+1}]} = k < 2^{\ell+1}$. Since $|B_\ell| \equiv 0$ (mod $2^{\ell+1}$) and $n = |V| \equiv 1(\text{mod } 2^{\ell+1})$, we must have $|A| = 1$. Thus θ has exactly one fixed point and all other orbits have length divisible by $2^{\ell+1}$. ∎

2.3 Necessary and sufficient conditions on order

In 2007, Szymański and Wojda [32] solved the problem of the existence of a self-complementary k-hypergraph of order n.

Theorem 2.3.1 [32] *Let k and n be positive integer such that $k \leq n$. A self-complementary k-uniform hypergraph of order n exists if and only if $\binom{n}{k}$ is even.*

In this section, we give an alternative description of the condition that $\binom{n}{k}$ is even in terms of the binary representation of k (see Corollary 2.3.2) which is more easily verified. In particular, this yields more transparent conditions on the order of a self-complementary k-hypergraph when the rank k is a sum of consecutive powers of 2.

Lemma 2.2.6(2) and Theorem 2.2.5 imply the following necessary and sufficient conditions on the order of a self-complementary uniform hypergraph of rank k.

Corollary 2.3.2
Let k and n be positive integers, $k \leq n$, and let b be the binary representation of k. There exists a self-complementary k-hypergraph of order n if and only if

$$n_{[2^{\ell+1}]} < k_{[2^{\ell+1}]} \text{ for some } \ell \in supp(b). \tag{2.3.1}$$

Proof: Suppose that there exists a self-complementary k-hypergraph $X = (V, E)$. Lemma 2.2.6(2) implies that there exists $\theta \in \mathrm{Ant}(X)$ with order equal to a power of 2. Thus Theorem 2.2.5 implies that we can partition V into disjoint sets A and B such that A and B are unions of orbits of θ, and there exists $\ell \in \mathrm{supp}(b)$ such $|A| < k_{[2^{\ell+1}]}$ and $|B| \equiv 0 \pmod{2^{\ell+1}}$. Since $n = |V| = |A| + |B|$, it follows that $n_{[2^{\ell+1}]} < k_{[2^{\ell+1}]}$. Hence (2.3.1) holds.

Conversely, suppose that (2.3.1) holds for some $\ell \in \mathrm{supp}(b)$, say $n = m2^{\ell+1} + j$ for some $j < k_{[2^{\ell+1}]}$. Let V be a set of order n, and let θ be a permutation in $Sym(V)$ which has j fixed points and m cycles of length $2^{\ell+1}$. Then θ satisfies the conditions of Theorem 2.2.5 for ℓ, and so θ is a k-complementing permutation. Thus there exists a self-complementary k-hypergraph of order n in the θ-switching class of self-complementary hypergraphs on V. ∎

In Appendix A, Lemma A.0.13, we show directly that condition (2.3.1) is equivalent to Szymański and Wojda's condition that $\binom{n}{k}$ is even.

When $k = 2^\ell$ or $k = 2^\ell + 1$, Corollary 2.3.2 yields the following result.

Corollary 2.3.3 *Let ℓ be a positive integer.*

1. *If $k = 2^\ell$, then there exists a self-complementary k-hypergraph of order n if and only if $n_{[2^{\ell+1}]} < k$.*

2. *If $k = 2^\ell + 1$, then there exists a self-complementary k-hypergraph of order n if and only if n is even or $n_{[2^{\ell+1}]} < k$.*

∎

For example, there exists a self-complementary graph of order n if and only if $n \equiv 0$ or $1 \pmod{4}$, and there exists a self-complementary 3-hypergraph of order n if and only if $n \equiv 0, 1$ or $2 \pmod{4}$. In the case where k is a sum of consecutive powers of 2, the condition of Corollary 2.3.2 holds for the largest integer in the support of the binary representation of k, as the next result shows.

Corollary 2.3.4 *Let r and ℓ be nonnegative integers, and suppose that $k = \sum_{i=0}^{r} 2^{\ell+i}$. Then there exists a self-complementary k-hypergraph of order n if and only if $n_{[2^{\ell+r+1}]} < k$.*

Proof: Suppose that there exists a self-complementary k-hypergraph of order n, and let b be the binary representation of k. Then

$$\mathrm{supp}(b) = \{\ell, \ell+1, \ldots, \ell+r\},$$

and so Corollary 2.3.2 guarantees that

$$n_{[2^{\ell+j+1}]} < k_{[2^{\ell+j+1}]} \tag{2.3.2}$$

for some $j \in \{0, 1, 2, \ldots, r\}$. If (2.3.2) holds for some $j < r$, then the fact that

$$n_{[2^{\ell+(j+1)+1}]} \leq 2^{\ell+j+1} + n_{[2^{\ell+j+1}]}$$

implies that

$$n_{[2^{\ell+(j+1)+1}]} < 2^{\ell+j+1} + k_{[2^{\ell+j+1}]}. \tag{2.3.3}$$

Now since $2^{\ell+j+1} + k_{[2^{\ell+j+1}]} = 2^{\ell+j+1} + \sum_{i=0}^{j} 2^{\ell+i} = k_{[2^{\ell+(j+1)+1}]}$, inequality (2.3.3) implies that

$$n_{[2^{\ell+(j+1)+1}]} < k_{[2^{\ell+(j+1)+1}]},$$

and hence (2.3.2) also holds for $j+1$. Thus, by induction on j, the fact that (2.3.2) holds for some $j \in \{0, 1, \ldots, r\}$ implies that (2.3.2) holds for $j = r$. Hence $n_{[2^{\ell+r+1}]} < k_{[2^{\ell+r+1}]} = k$.

Conversely, Corollary 2.3.2 guarantees that there exists a self-complementary k-hypergraph of order n for every integer n such that $n_{[2^{\ell+r+1}]} < k_{[2^{\ell+r+1}]} = k$. ∎

Corollary 2.3.5 *Let ℓ be a positive integer and suppose that $k = 2^\ell - 1$.*

(1) There exists a self-complementary k-hypergraph of order n if and only if $n_{[2^\ell]} < k$.

(2) If $n = 2^r - 1$ for some integer $r \geq \ell$, there does not exist a self-complementary k-hypergraph of order n.

Proof: Since $k = 2^\ell - 1 = \sum_{i=0}^{\ell-1} 2^i$, (1) follows directly from Corollary 2.3.4. If $n = 2^r - 1$ for $r \geq \ell$, then $n_{[2^\ell]} = 2^\ell - 1 = k$, and so (1) implies that there does not exist a self-complementary k-hypergraph of order n. Hence (2) holds. ∎

Corollary 2.3.6 *If $n = 2^r - 2$ for an integer $r \geq 2$, then there exists a self-complementary k-hypergraph of order n if and only if $k < n$ and k is odd.*

Proof: Let b denote the binary representation of k. If k is odd, then $0 \in \text{supp}(b)$. Since n is even, Corollary 2.3.2 implies that there exists a self-complementary k-hypergraph of order n.

Conversely, suppose that k is even and $k < n$. Then since $k < n$, $\max\{\ell : \ell \in \text{supp}(b)\} \leq r$, and so $n_{[2^\ell]} = 2^\ell - 2$ for all $\ell \in \text{supp}(b)$. Since k is even, $2^\ell - 2 \geq k_{[2^\ell]}$, and so $n_{[2^\ell]} \geq k_{[2^\ell]}$ for all $\ell \in \text{supp}(b)$. Thus Corollary 2.3.2 implies that there does not exist a self-complementary k-hypergraph of order n. ∎

2.4 Generating self-complementary hypergraphs

In this section we present Algorithm 2.4.3, which generates all self-complementary k-hypergraphs of order n, up to isomorphism, and Algorithm 2.4.4, which determines whether a given permutation in $Sym(n)$ is a k-complementing permutation. Before we present these algorithms, we will need some terminology and a couple of preliminary algorithms and results.

If $\theta \in Sym(n)$ is the product of disjoint cycles of lengths $n_1, n_2, \ldots, n_r$ with $n_1 \leq n_2 \leq \cdots \leq n_r$ (including 1-cycles), then the r-tuple $(n_1, n_2, \ldots, n_r)$ is called the *cycle type* of θ. An *integer partition* of a positive integer n is a list of positive integers $(n_1, n_2, \ldots, n_r)$ such that $n_1 \leq n_2 \leq \cdots \leq n_r$, and $\sum_{i=1}^{r} n_i = n$. It is well known that two permutations in $Sym(n)$ are conjugate if and only if they have the same cycle type. Hence there is a natural correspondence between the conjugacy classes of $Sym(n)$, the cycle types of $Sym(n)$, and the integer partitions of n.

Given an integer partition $p = (n_1, n_2, \ldots, n_r)$ of n, let $\theta(p, n)$ denote the permutation in $Sym(n)$ with cycle type p, whose i-th cycle has the j-th entry equal to $\sum_{t=0}^{i-1} n_i + j$, for all $i \in \{1, 2, \ldots, r\}$ and all $j \in \{1, 2, \ldots, n_i\}$, where $n_0 = 0$. For example, for the partition $p = (2, 3, 3)$ of $n = 8$, we have $\theta(p, n) = (1\ 2)(3\ 4\ 5)(6\ 7\ 8)$.

Recall that for each k-complementing permutation θ in $Sym(V)$, Algorithm 2.1.2 will generate the set $\mathcal{H}_\theta$ of all self-complementary k-hypergraphs on V for which θ is an antimorphism, up to isomorphism. Lemma 2.2.6(2) guarantees that every self-complementary k-hypergraph has an antimorphism which has order a power of 2, and so we can generate all of the self-complementary k-hypergraphs of order n, up to isomorphism, by applying Algorithm 2.1.2 to each permutation in $Sym(n)$ satisfying the conditions of Theorem 2.2.5. However, Lemma 2.4.1 below shows that we need only apply Algorithm 2.1.2 to a set of conjugacy class representatives of such permutations.

Lemma 2.4.1 *Two permutations θ and σ are conjugate in $Sym(V)$ if and only if each hypergraph in $\mathcal{H}_\theta$ is isomorphic to a hypergraph in $\mathcal{H}_\sigma$.*

Proof: Observe that if θ is an antimorphism of a k-hypergraph $X = (V, E)$, and $\sigma = \tau^{-1}\theta\tau$ is conjugate to θ, then σ is an antimorphism of $X^\tau = (V, E^\tau)$. Hence each hypergraph X in $\mathcal{H}_\theta$ is isomorphic to a hypergraph X^τ in H_σ. Conversely, if X and Y are isomorphic self-complementary k-hypergraphs, say $X^\tau = Y$, then if $\theta \in Ant(X)$ it follows that $\tau^{-1}\theta\tau \in Ant(Y)$, and so any two isomorphic self-complementary k-hypergraphs have antimorphisms from the same conjugacy class. ∎

Lemma 2.4.1 implies that in order to generate all of the self-complementary k-

hypergraphs of order n up to isomorphism, it suffices to apply Algorithm 2.1.2 to one permutation from each conjugacy class of permutations in $Sym(n)$ satisfying the conditions of Theorem 2.2.5. We will do this in Algorithm 2.4.3. First, we will need some notation and a preliminary algorithm.

For a positive integer n, let $\mathcal{P}_2(n)$ denote the set of integer partitions of n into powers of 2. Note that a complete set of representatives of the conjugacy classes of permutations in $Sym(n)$ satisfying the conditions of Theorem 2.2.5 corresponds to a subset of $\mathcal{P}_2(n)$, and so we will make use of the set $\mathcal{P}_2(n)$ in Algorithm 2.4.3. In [15], Riha and James present an efficient algorithm for generating the set of integer partitions of n with a fixed number m of parts from a given set S. This algorithm can be applied with m parts from $S = \{2^i : 2^i \leq n\}$, for each $m = 1, 2, \ldots, n$, to construct the set $\mathcal{P}_2(n)$.

In Algorithm 2.4.2, we will describe an alternative method for generating the set $\mathcal{P}_2(n)$ for any positive integer n. For a positive integer r and two vectors $w, v \in \mathbb{Z}^r$, we write $w \leq v$ if w is less than or equal to v with respect to the lexicographic, or dictionary, ordering. We will use $sort(v)$ to denote the vector obtained from v by sorting its coordinates in non-decreasing order, and $concatenate(v, w)$ to denote the vector $(v_1, v_2, \ldots, v_r, w_1, w_2, \ldots, w_r) \in \mathbb{Z}^{2r}$ obtained by concatenating the vectors v and w. Note that $sort(v)$ can be obtained from v using any of the well-known sorting algorithms, such as the *Quicksort* algorithm developed by Hoare in [13].

Algorithm 2.4.2 Let n be a positive integer and let c be the binary representation of n. Let $supp(c) = \{\ell_1, \ell_2, \ldots, \ell_t\}$, where $\ell_1 < \ell_2 < \cdots < \ell_t$.

(1) Using steps (A) and (B) below recursively, construct the sets $\mathcal{P}_2(2^i)$ recursively for $i = 0, 1, \ldots, \ell_t$.

  (A) Set $\mathcal{P}_2(2^0) := \{(1)\}$. Set $i := 1$.

  (B) Repeat steps (I)-(III) below while $i \leq \ell_t$.

(I) Set $\mathcal{P} := \mathcal{P}_2(2^{i-1}) \times \mathcal{P}_2(2^{i-1})$. Set $\mathcal{P}_2(2^i) := \{(2^i)\}$.

Repeat steps (a)-(b) below while $\mathcal{P} \neq \emptyset$.

(a) Choose $(p, q) \in \mathcal{P}$. If $p \leq q$, then set $\mathcal{P}_2(2^i) := \mathcal{P}_2(2^i) \cup \{sort(concatenate(p, q))\}$.

(b) Set $\mathcal{P} := \mathcal{P} \setminus \{(p, q)\}$. Return to step (a).

(II) Return $\mathcal{P}_2(2^i)$.

(III) Set $i := i + 1$. Return to step (I).

(2) Using steps (A),(B), and (C) below, construct the set $\mathcal{P}_2(n)$.

(A) Set $\mathcal{P} := \mathcal{P}_2(2^{\ell_1}) \times \mathcal{P}_2(2^{\ell_2}) \times \cdots \times \mathcal{P}_2(2^{\ell_t})$. Set $\mathcal{P}_2(n) := \emptyset$.

(B) Repeat steps (I)-(II) below while $\mathcal{P} \neq \emptyset$.

(I) Choose $(p_1, p_2, \ldots, p_t) \in \mathcal{P}$.
Set $\mathcal{P}_2(n) := \mathcal{P}_2(n) \cup \{sort(concatenate(p_1, p_2, \ldots, p_t))\}$.

(II) Set $\mathcal{P} := \mathcal{P} \setminus \{(p_1, p_2, \ldots, p_t)\}$. Return to step (I).

(C) Return $\mathcal{P}_2(n)$.

Lemma A.0.14 guarantees that every element in $\mathcal{P}_2(2^i)$ (except for (2^i) itself) is the concatenation of two elements in $\mathcal{P}_2(2^{i-1})$. Hence steps (1)(A)-(1)(B) of Algorithm 2.4.2 generate all of $\mathcal{P}_2(2^i)$, for $i = 1, 2, \ldots, \ell_t$. Also, the uniqueness of the binary representation of n and Lemma A.0.14 together guarantee that every element if $\mathcal{P}_2(n)$ is the concatenation of the coordinates of an element in $\mathcal{P}_2(2^{\ell_1}) \times \mathcal{P}_2(2^{\ell_2}) \times \ldots \times P_2(2^{\ell_t})$. Hence steps (2)(A)-(2)(C) of Algorithm 2.4.2 generate all of $\mathcal{P}_2(n)$.

We are ready to state Algorithm 2.4.3, which generates all self-complementary k-hypergraphs of order n, up to isomorphism, without prior input of a k-complementing permutation in $Sym(n)$.

Algorithm 2.4.3 Let n and k be positive integers, $k \leq n$. Let b be the binary representation of k, and let

$$\mathcal{L} := \{\ell \in \text{supp}(b) : n_{[2^{\ell+1}]} < k_{[2^{\ell+1}]}\}.$$

(1) Set $\mathcal{H} := \emptyset$. If $\mathcal{L} = \emptyset$, go to step (4). Otherwise, go to step (2).

(2) Construct the set $\mathcal{P}_2(n)$ using Algorithm 2.4.2.
Set $\mathcal{P} := \mathcal{P}_2(n)$.
Using steps (A) and (B) below, construct a set S of representatives of the conjugacy classes of permutations in $Sym(n)$ which satisfy the conditions of Theorem 2.2.5 for some $\ell \in \text{supp}(b)$.

(A) Set $S := \emptyset$. Repeat steps (a)-(b) below while $\mathcal{P} \neq \emptyset$.

(a) Choose $p = (n_1, n_2, \ldots, n_r) \in \mathcal{P}$ and set

$$L_p := \{\ell \in \mathcal{L} : n_i \neq 2^\ell \text{ for } 1 \leq i \leq r\}.$$

Repeat step (i) below while $L_p \neq \emptyset$.

(i) Choose $\ell \in L_p$. Let $n_0 = 0$, and let s be the largest integer such that $n_i < 2^\ell$ for all $i \leq s$.
If $\sum_{i=0}^{s} n_i < \sum_{i=0}^{\ell} b_i 2^i$, set $S := S \cup \{\theta(p, n)\}$ and set $L_p = \emptyset$.
Otherwise, set $L_p := L_p \setminus \{\ell\}$.

(b) Set $\mathcal{P} := \mathcal{P} \setminus \{p\}$. Return to step (a).

(B) Return S.

(3) Repeat step (A) below while $S \neq \emptyset$.

(A) Choose $\theta \in S$ and apply Algorithm 2.1.2 to construct $\mathcal{H}_\theta$.
Set $\mathcal{H} := \mathcal{H} \cup \mathcal{H}_\theta$.
Set $S := S \setminus \{\theta\}$.

(4) Return $\mathcal{H}$.

The following algorithm determines whether or not a permutation $\theta \in Sym(V)$ is a k-complementing permutation.

Algorithm 2.4.4 Let k be a positive integer, let b be the binary representation of k, and let V be a finite set.

Input: $\theta \in Sym(V)$

$$\text{Output : } \begin{cases} \text{YES}, & \text{if } \theta \text{ is a } k\text{-complementing permutation} \\ \text{NO}, & \text{otherwise} \end{cases} .$$

1. If $|\theta|$ is odd, output NO and quit.

   Otherwise, go to step (2).

2. Write $|\theta| = 2^i(2t+1)$ for some positive integer i.

   Let $\hat{\theta} = \theta^{2t+1}$, and let $p = (n_1, n_2, \ldots, n_r)$ be the cycle type of $\hat{\theta}$. Set

   $$L_p := \{\ell \in \text{supp}(b) : n_i \neq 2^\ell \text{ for all } i \in \{1, 2, \ldots, r\}\}.$$

   If $L_p = \emptyset$, output NO and quit.

   Otherwise, go to step (3).

3. Choose $\ell \in L_p$.

   Let $n_0 = 0$, and let s be the largest integer such that $n_i < 2^\ell$ for all $i \leq s$. If $\sum_{i=0}^{s} n_i < \sum_{i=0}^{\ell} b_i 2^i$, output YES and quit.

   Otherwise, go to step (4).

4. Set $L_p := L_p \setminus \{\ell\}$. If $L_p = \emptyset$, output NO and quit.

   Otherwise, return to step (3).

The method of Algorithm 2.4.4 for testing whether θ is a k-complementing permutation relies on the characterization of k-complementing permutations given in Corollary 2.2.7, which is an alternative to Wojda's characterization in Theorem 2.2.4.

Chapter 3

Regular self-complementary hypergraphs

In this chapter, we examine the orders of t-subset-regular self-complementary k-uniform hypergraphs.

3.1 Necessary conditions on order

We find necessary conditions on the order of a t-subset-regular self-complementary k-hypergraph in terms of the binary representation of the rank k. In Section 3.1.1 we will present the results to date, and in Section 3.1.2 we will reformulate the previously obtained necessary conditions on the order of these structures in terms of the binary representation of the rank k.

3.1.1 Previous results

We state the known necessary conditions on the order of t-subset-regular self-complementary k-hypergraphs.

Clearly, if a self-complementary k-hypergraph $X = (V, E)$ of order $|V| = n$ exists, then $\{X, X^C\}$ is a partition of $V^{(k)}$ into two sets of equal size $|V^{(k)}|/2 = \binom{n}{k}/2$, and consequently $\binom{n}{k}$ must be even. If X is also t-subset-regular, we obtain additional necessary conditions, which are given in Theorem 3.1.1. This result was first stated by Hartman [12] in the language of large sets of t-designs. The proof is included here for the sake of completeness.

Theorem 3.1.1 [12] *Suppose X is a t-subset-regular self-complementary k-hypergraph of order n, where $0 < t < k < n$. Then*

$$\binom{n-i}{k-i} \equiv 0 \pmod 2 \quad \textit{for } 0 \le i \le t. \tag{3.1.1}$$

Proof: Let T be any t-subset of $V = V(X)$. Clearly T is contained in exactly $\binom{n-t}{k-t}$ sets in $V^{(k)}$. Since X is self-complementary, the t-valency of X is equal to the t-valency of X^C, and these t-valencies must sum to $\binom{n-t}{k-t}$. Hence the t-valency of X is $\binom{n-t}{k-t}/2$.

Now let $i \in \{0, 1, \ldots, t\}$, and let I be any i-subset of V. We will count the number of edges of X containing I. We know that the i-subset I is contained in exactly $\binom{n-i}{t-i}$ t-subsets of V, and each t-subset of V is contained in exactly $\binom{n-t}{k-t}/2$ edges of X. Since each edge of X contains $\binom{k-i}{t-i}$ t-subsets which contain the i-subset I, it follows that I lies in exactly

$$\frac{1}{2}\binom{n-t}{k-t}\frac{\binom{n-i}{t-i}}{\binom{k-i}{t-i}} = \frac{1}{2}\binom{n-i}{k-i}$$

edges of X, which is independent of the choice of I. Hence X is i-subset-regular, with i-valency $\binom{n-i}{k-i}/2$. Since this number must be an integer, and i was chosen arbitrarily, we conclude that the (3.1.1) holds. ∎

The proof of Theorem 3.1.1 actually shows that a t-subset-regular self-complementary hypergraph is necessarily i-subset-regular, for all positive integers $i \le t$.

The necessary conditions (3.1.1) of Theorem 3.1.1 are stated alternatively in Theorem 3.1.2. The equivalence of these two statements was proved by Khosrovshahi and Tayfeh-Rezaie [17], also in the language of large sets of t-designs.

Theorem 3.1.2 [17] *Let t, k and n be positive integers such that $t < k \leq n$. If there exists a t-subset-regular self-complementary k-hypergraph of order n, then there exists a positive integer a such that* $\max\{i : 2^i \mid k\} < a \leq \min\{i : 2^i > k\}$ *and*

$$n_{[2^a]} \in \{t, t+1, \ldots, k_{[2^a]} - 1\}. \tag{3.1.2}$$

3.1.2 New results

In this section, we will reformulate the known necessary conditions on the order of a t-subset-regular self-complementary k-hypergraph in terms of the binary representation of the rank k. This yields more transparent conditions on the order n in the case where k is a sum of consecutive powers of 2.

In Theorem 3.1.4, we refine the result of Theorem 3.1.2 slightly to show that (3.1.2) holds for an integer a such that $a-1$ lies in the support of the binary representation of k. It should be noted that Potočnik and Šajna first observed this refinement in the case where the rank k has the form $k = 2^\ell$ or $k = 2^\ell + 1$ [24].

First we need a preliminary lemma.

Lemma 3.1.3 *Let t, k and n be positive integers such that $t < k \leq n$. Let b be the binary representation of k. There exists a positive integer a such that* $\max\{i : 2^i \mid k\} < a \leq \min\{i : 2^i > k\}$ *and*

$$n_{[2^a]} \in \{t, t+1, \ldots, k_{[2^a]} - 1\} \tag{3.1.3}$$

if and only if there exists $\ell \in supp(b)$ such that (3.1.3) holds with $a = \ell + 1$.

Proof: ($\Leftarrow$) If $\ell \in \operatorname{supp}(b)$ then $\max\{i : 2^i \mid k\} < \ell + 1 \leq \min\{i : 2^i > k\}$. Hence if (3.1.3) holds with $a = \ell + 1$ for some $\ell \in \operatorname{supp}(b)$, then certainly (3.1.3) holds for an integer a in the range $\max\{i : 2^i \mid k\} < \ell + 1 \leq \min\{i : 2^i > k\}$.

($\Rightarrow$) Suppose that there exists an integer a such that $\max\{i : 2^i \mid k\} < a \leq \min\{i : 2^i > k\}$ and (3.1.3) holds. If $a - 1 \in \mathrm{supp}(b)$, then set $\ell = a - 1$ and we are done. Hence we may assume that $a - 1 \notin \mathrm{supp}(b)$.

Now if $i \notin \mathrm{supp}(b)$ for all i such that $0 \leq i \leq a - 1$, then $k_{[2^a]} = \sum_{i=0}^{a-1} b_i 2^i = 0$, and so as $t \geq 1$, (3.1.3) implies that $n_{[2^a]} \in \emptyset$, giving a contradiction. Hence we must have $\mathrm{supp}(b) \cap \{1, 2, \dots, a - 1\} \neq \emptyset$. Set

$$\ell = \max\{i : i \in \mathrm{supp}(b) \cap \{1, 2, \dots, a - 1\}\}$$

and observe that $\ell < a - 1$. Then $k_{[2^a]} = \sum_{i=0}^{a-1} b_i 2^i = \sum_{i=0}^{\ell} b_i 2^i = k_{[2^{\ell+1}]}$, and so (3.1.3) implies that

$$n_{[2^a]} \in \{t, t + 1, \dots, k_{[2^{\ell+1}]} - 1\}. \tag{3.1.4}$$

Now (3.1.4) implies that $n_{[2^a]} < 2^{\ell+1}$. Since $\ell + 1 < a$, it follows that $n_{[2^{\ell+1}]} = n_{[2^a]}$, and so (3.1.4) implies that

$$n_{[2^{\ell+1}]} \in \{t, t + 1, \dots, k_{[2^{\ell+1}]} - 1\}.$$

Since $\ell \in \mathrm{supp}(b)$, this completes the proof. ∎

Theorem 3.1.4 *Let k be a positive integer and let $b = (b_m, b_{m-1}, \dots, b_2, b_1, b_0)_2$ be the binary representation of k. Let t be an integer such that $1 \leq t < k$. If there exists a t-subset-regular self-complementary k-hypergraph, then*

$$n_{[2^{\ell+1}]} \in \{t, t + 1, \dots, k_{[2^{\ell+1}]} - 1\} \tag{3.1.5}$$

for some $\ell \in supp(b)$.

Proof: If there exists a t-subset-regular self-complementary k-hypergraph, then by Theorem 3.1.2, there exists a positive integer a such that (3.1.3) holds. Hence (3.1.5) follows from Lemma 3.1.3. ∎

Corollary 3.1.5 *Let ℓ be a positive integer, let $k = 2^\ell$ or $k = 2^\ell + 1$, and let t be an integer such that $1 \leq t < k$. If there exists a t-subset-regular self-complementary k-hypergraph of order n, then $n_{[2^{\ell+1}]} \in \{t, t+1, \ldots, k-1\}$.*

Proof: If there exists a t-subset-regular self-complementary k-hypergraph of order n, then Theorem 3.1.4 implies that condition (3.1.5) holds for some $\hat{\ell}$ in the support of the binary representation b of k. Since $t \geq 1$, for $\hat{\ell} = 0$ we have $\{t, t+1, \ldots, k_{[2^{\hat{\ell}+1}]} - 1\} = \emptyset$. Hence condition (3.1.5) must hold for some nonzero element of supp(b). Since $k \in \{2^\ell, 2^\ell + 1\}$, the only nonzero element in supp(b) is ℓ. Hence (3.1.5) holds for ℓ, and so $n_{[2^{\ell+1}]} \in \{t, t+1, \ldots, k_{[2^{\ell+1}]} - 1\}$. Since $k_{[2^{\ell+1}]} = k$ for all $k \in \{2^\ell, 2^\ell + 1\}$, the result follows. ∎

In the case where k is a sum of consecutive powers of 2, if condition (3.1.5) of Theorem 3.1.4 holds, then it holds for the largest integer in the support of the binary representation of k, as the next corollary shows.

Corollary 3.1.6 *Let r and ℓ be nonnegative integers, and suppose that $k = \sum_{i=0}^{r} 2^{\ell+i}$. If there exists a t-subset-regular self-complementary k-hypergraph of order n, then $n_{[2^{\ell+r+1}]} \in \{t, t+1, \ldots, k-1\}$.*

Proof: Let b denote the binary representation of k. Then

$$\text{supp}(b) = \{\ell, \ell+1, \ldots, \ell+r\},$$

and so Theorem 3.1.4 guarantees that

$$n_{[2^{\ell+j+1}]} \in \{t, t+1, \ldots, k_{[2^{\ell+j+1}]} - 1\} \tag{3.1.6}$$

for some $j \in \{0, 1, \ldots, r\}$. Suppose that $j < r$. Since

$$n_{[2^{\ell+(j+1)+1}]} \leq 2^{\ell+j+1} + n_{[2^{\ell+j+1}]}$$

and

$$n_{[2^{\ell+(j+1)+1}]} \geq n_{[2^{\ell+j+1}]},$$

condition (3.1.6) implies that

$$t \leq n_{[2^{\ell+(j+1)+1}]} < 2^{\ell+j+1} + k_{[2^{\ell+j+1}]}. \tag{3.1.7}$$

Now since $2^{\ell+j+1} + k_{[2^{\ell+j+1}]} = 2^{\ell+j+1} + \sum_{i=0}^{j} 2^{\ell+i} = k_{[2^{\ell+(j+1)+1}]}$, inequalities (3.1.7) imply that

$$t \leq n_{[2^{\ell+(j+1)+1}]} < k_{[2^{\ell+(j+1)+1}]},$$

and hence $n_{[2^{\ell+(j+1)+1}]} \in \{t, t+1, \ldots, k_{[2^{\ell+(j+1)+1}]} - 1\}$. Thus for $j < r$, we have that

$$n_{[2^{\ell+j+1}]} \in \{t, t+1, \ldots, k_{[2^{\ell+j+1}]} - 1\}$$

implies

$$n_{[2^{\ell+(j+1)+1}]} \in \{t, t+1, \ldots, k_{[2^{\ell+(j+1)+1}]} - 1\}.$$

It follows that

$$n_{[2^{\ell+r+1}]} \in \{t, t+1, \ldots, k_{[2^{\ell+r+1}]} - 1\}.$$

Since $k_{[2^{\ell+r+1}]} = k$, this implies that

$$n_{[2^{\ell+r+1}]} \in \{t, t+1, \ldots, k-1\}$$

as claimed. ∎

Corollary 3.1.7 *Let ℓ be a positive integer, let $k = 2^{\ell} - 1$ and let t be a positive integer such that $t < k$. If there exists a t-subset-regular self-complementary k-hypergraph of order n, then $n_{[2^{\ell}]} \in \{t, t+1, \ldots, k-1\}$.*

Proof: Since $k = 2^\ell - 1 = \sum_{i=0}^{\ell-1} 2^i$, this result follows directly from Corollary 3.1.6. ∎

3.2 Sufficient conditions on order

In this section, we show that the necessary conditions on the order of a t-subset-regular self-complementary k-hypergraph given in Theorem 3.1.2 are sufficient in certain cases.

3.2.1 Previous results

The necessary conditions of Theorem 3.1.2 have been shown to be sufficient for all t in the cases where $k \in \{2, 3\}$. Rao handled the case where $k = 2$ [27], Potočnik and Šajna handled the case where $k = 3$ and $t = 1$ [23], and Knor and Potočnik handled the case where $k = 3$ and $t = 2$ [18]. Thus the condition of Theorem 3.1.2 is both necessary and sufficient when $k \in \{2, 3\}$, as the following result states.

Theorem 3.2.1 *Let n be a positive integer.*

(1) [27] There exists a regular self-complementary graph of order n if and only if n is congruent to 1 modulo 4.

(2) [23] There exists a 1-subset-regular self-complementary 3-hypergraph of order n if and only if n is congruent to 1 or 2 modulo 4.

(3) [18] There exists a 2-subset-regular self-complementary 3-hypergraph of order n if and only if n is congruent to 2 modulo 4.

3.2.2 New results

In this section, we prove that a 1-subset-regular self-complementary k-hypergraph of order n exists for every integer n satisfying the necessary conditions of Theorem 3.1.2. First we will need some notation.

In this section, we will denote the vertex set and edge set of a self-complementary k-hypergraph X by $\mathcal{V}(X)$ and $\mathcal{E}(X)$, respectively. Also, we will denote the valency $val_X^1(\{v\})$ defined on page 3 of Section 1.1 by $val_X(v)$. Let $X = (\mathcal{V}, \mathcal{E})$ be a k-hypergraph and let $\theta \in Sym(\mathcal{V})$. Then X^θ denotes the hypergraph $(\mathcal{V}, \mathcal{E}^\theta)$, where $\mathcal{E}^\theta = \{E^\theta : E \in \mathcal{E}\}$ and $E^\theta = \{v^\theta : v \in E\}$. For a subset $\mathcal{P}$ of the orbits of θ on $\mathcal{V}^{(k)}$, let $\mathcal{U}(\mathcal{P}) = \bigcup_{\mathcal{O} \in \mathcal{P}} \mathcal{O}$. For a subset $\mathcal{S} \subseteq \mathcal{V}^{(k)}$ and a vertex $v \in \mathcal{V}$, let $setval_{\mathcal{S}}(v)$ denote the number of edges of $\mathcal{S}$ containing v, and let $\mathcal{S}^C$ denote the complement of $\mathcal{S}$ in $\mathcal{V}^{(k)}$. That is, $\mathcal{S}^C = \mathcal{V}^{(k)} \setminus \mathcal{S}$.

We will often make use of the following lemma.

Lemma 3.2.2 *Suppose that $X = (\mathcal{V}, \mathcal{E})$ is a self-complementary k-hypergraph.*

(1) X is 1-subset-regular if and only if $val_X(v) = val_{X^C}(v)$ for all $v \in \mathcal{V}$.

(2) If $\mathcal{V} = \{\infty\} \cup \mathbb{Z}_n$ and $\theta = (\infty)(0\ 1\ \ldots\ (n-1)) \in Ant(X)$ for an even positive integer n, then X is 1-subset-regular if and only if $val_X(0) = val_{X^C}(0)$.

Proof:

(1) If $val_X(v) = val_{X^C}(v)$, then

$$val_X(v) = \frac{1}{2}\left(val_X(v) + val_{X^C}(v)\right) = \frac{1}{2}\left(setval_{\mathcal{V}^{(k)}}(v)\right) = \frac{1}{2}\binom{|\mathcal{V}|-1}{k-1},$$

which is independent of the choice of $v \in \mathcal{V}$. Thus X is 1-subset-regular. Conversely, if X is 1-subset-regular, then since $X \cong X^C$, the hypergraph X^C is also 1-subset-regular and has the same 1-valency as X. Thus $val_X(v) = val_{X^C}(v)$ for all $v \in \mathcal{V}$.

(2) If X is 1-subset-regular, then as $0 \in \mathcal{V}$, $val_X(0) = val_{X^C}(0)$ by part (1).

Conversely, suppose that $val_X(0) = val_{X^C}(0)$. Observe that for any orbit $\mathcal{O}$ of θ on $\mathcal{V}^{(k)}$, an element $x \in \mathbb{Z}_n$ lies in c edges of $\mathcal{O} \cap \mathcal{E}$ if and only if $(x-1)_{[n]}$ lies in c edges of $\mathcal{O} \cap \mathcal{E}^C$, which holds if and only if $(x-2)_{[n]}$ lies in c edges of $\mathcal{O} \cap \mathcal{E}$. This implies that $val_X(x) = val_{X^C}(y)$ whenever $x \not\equiv y \pmod 2$, and $val_X(x) = val_X(y)$ whenever $x \equiv y \pmod 2$. Now since $val_X(0) = val_{X^C}(0)$, for x even and y odd, we have

$$val_X(x) = val_X(0) = val_{X^C}(0) = val_X(y).$$

Hence $val_X(x) = val_X(y)$ for all $x, y \in \mathbb{Z}_n$. Moreover, since $\theta \in \mathrm{Ant}(X)$ and θ fixes ∞, we must also have $val_X(\infty) = val_{X^C}(\infty)$. Hence $val_X(v)$ is independent of the choice of $v \in \mathcal{V}$, and so X is 1-subset-regular.

∎

In Lemma 3.2.5 we will prove that the necessary condition (3.1.3) in Theorem 3.1.2 on the order n of a self-complementary k-hypergraph is sufficient by induction on the congruence class of n modulo 2^a. In Lemma 3.2.4 we will handle the base case where $n \equiv 1 \pmod{2^a}$, that is, $n = m2^a + 1$ for some positive integer m. We will need to make use of the following lemma, which handles the case where $m = 1$.

For a positive integer n, a subset A of $\mathbb{Z}_n$, and an element b of $\mathbb{Z}_n$, let $A + b$ denote the set $\{(a+b)_{[n]} : a \in A\}$.

Lemma 3.2.3 *Let k and a be positive integers such that $a \geq 2$ and $2 \leq k < 2^a$. Let $\mathcal{V} = \{\infty\} \cup \mathbb{Z}_{2^a}$, where $\infty \notin \mathbb{Z}_{2^a}$. There exists a 1-subset-regular self-complementary k-hypergraph on $\mathcal{V}$ with antimorphism*

$$\theta = (\infty)(0\ 1\ 2\ \cdots\ (2^a - 1)).$$

Proof: First we will fix an integer r such that $1 \leq r < 2^a$ and examine the structure of the orbits of θ on $\mathbb{Z}_{2^a}^{(r)}$. In particular, we will examine how the number of even and odd elements of $E \in \mathbb{Z}_{2^a}^{(r)}$ affect the valency of 0 in the orbit of θ on $\mathbb{Z}_{2^a}^{(r)}$ containing E.

We can write $r = 2^z M$ for some integer z such that $0 \leq z \leq a-1$ and some odd positive integer M. Let $\mathcal{O}$ be an orbit of θ on $\mathbb{Z}_{2^a}^{(r)}$. Then $\mathcal{O}$ has length 2^{a-x} for some x such that $0 \leq x \leq z$. For $x \in \{0, 1, \ldots, z\}$, we will define a partition of the set $\{0, 1, 2, \ldots, 2^a - 1\}$ into 2^x subsets $S_0^x, S_1^x, S_2^x, \ldots, S_{2^x-1}^x$ of consecutive integers, each of length 2^{a-x}. For each $w \in \{0, 1, \ldots, 2^x - 1\}$, set $S_w^x := \{w2^{a-x} + v : v \in \{0, 1, \ldots, 2^{a-x} - 1\}\}$. Then $S_w^x = S_0^x + w2^{a-x}$. If $\mathcal{O}$ has length 2^{a-x}, any edge $E \in \mathcal{O}$ contains exactly $r/2^x$ elements from each subset S_w^x in the partition, and $E \cap S_w^x$ must be a translation of $E \cap S_0^x$, for all $w = 0, 1, \ldots, 2^x - 1$. In particular $E \cap S_w^x = E \cap S_0^x + w2^{a-x}$. Now if $E \cap S_0^x$ contains i even elements and j odd elements, then $E^\theta \cap S_0^x$ contains j even elements and i odd elements. Hence, for any orbit $\mathcal{O}$ of θ on $\mathbb{Z}_{2^a}^{(r)}$ of length 2^{a-x}, there exist nonnegative integers i and j such that $i + j = r/2^x$, and every edge of $\mathcal{O}$ contains exactly i even elements and j odd elements of S_0^x, or vice versa. Moreover, if $E \in \mathcal{O}$ and E contains i even and j odd elements of S_0^x, then 0 lies in exactly i elements of the sequence $E, E^{\theta^2}, E^{\theta^4}, \ldots, E^{\theta^{2^{a-x}-2}}$, and 0 lies in exactly $j = r/2^x - i$ elements of the sequence $E^{\theta^1}, E^{\theta^3}, E^{\theta^5}, \ldots, E^{\theta^{2^{a-x}-1}}$.

Now for $x \in \{0, 1, \ldots, z\}$ and $i \in \{0, 1, \ldots, r/2^{x+1}\}$, and $j = r/2^x - i$, let $\mathcal{E}_{i,j}^x$ denote the set of orbits of θ on $\mathbb{Z}_{2^a}^{(r)}$ of length 2^{a-x} whose edges contain i even and j odd elements in the set S_0^x, or which contain j even and i odd elements of S_0^x. For each $\mathcal{O} \in \mathcal{E}_{i,j}^x$, choose an edge $E \in \mathcal{O}$. If the number of even elements of E does not exceed the number of odd elements of E, colour the edges in the sequence $E, E^{\theta^2}, E^{\theta^4}, \ldots, E^{\theta^{2^{a-x}-2}}$ red and colour the edges in the sequence $E^{\theta^1}, E^{\theta^3}, E^{\theta^5}, \ldots, E^{\theta^{2^{a-x}-1}}$ blue. If E has more even entries than odd entries, colour the edges in the sequence $E, E^{\theta^2}, E^{\theta^4}, \ldots, E^{\theta^{2^{a-x}-2}}$ blue and colour the edges in the sequence $E^{\theta^1}, E^{\theta^3}, E^{\theta^5}, \ldots, E^{\theta^{2^{a-x}-1}}$ red. For any subset $\mathcal{S}$ of $\mathbb{Z}_{2^a}^{(r)}$, let $\mathcal{S}_{red}$ and $\mathcal{S}_{blue}$

denote the set of red and blue edges in $\mathcal{S}$, respectively.

Now since $i \leq j$, for each orbit $\mathcal{O} \in \mathcal{E}_{i,j}^x$ we have

$$setval_{\mathcal{O}_{blue}}(0) - setval_{\mathcal{O}_{red}}(0) = j - i.$$

Let $\hat{\mathcal{E}}_r$ be a subset of $\mathbb{Z}_{2^a}^{(r)}$ which contains the red edges from exactly $\lfloor |\mathcal{E}_{i,j}^x| /2 \rfloor$ orbits of $\mathcal{E}_{i,j}^x$, and the blue edges from the remaining orbits of $\mathcal{E}_{i,j}^x$, for all $0 \leq x \leq z$, and for all i, j such that $0 \leq i \leq r/2^{x+1}$ and $j = r/2^x - i$. If $|\mathcal{E}_{i,j}^x|$ is even, say $|\mathcal{E}_{i,j}^x| = 2\nu$ for a positive integer ν, then

$$setval_{\mathcal{U}(\mathcal{E}_{i,j}^x \cap \hat{\mathcal{E}}_r)}(0) - setval_{\mathcal{U}(\mathcal{E}_{i,j}^x \cap \hat{\mathcal{E}}_r^C)}(0) = (\nu i + \nu j) - (\nu i + \nu j) = 0.$$

If $|\mathcal{E}_{i,j}^x|$ is odd, say $|\mathcal{E}_{i,j}^x| = 2\nu - 1$ for a positive integer ν, then

$$\begin{aligned} & setval_{\mathcal{U}(\mathcal{E}_{i,j}^x \cap \hat{\mathcal{E}}_r)}(0) - setval_{\mathcal{U}(\mathcal{E}_{i,j}^x \cap \hat{\mathcal{E}}_r^C)}(0) \\ =& ((\nu - 1)i + \nu j) - (\nu i + (\nu - 1)j) \\ =& j - i. \end{aligned}$$

Now if $x < z$, then $i + j = r/2^x = 2^{z-x}M$ is even, which implies that $j - i$ is even. On the other hand, if $x = z$, then $i + j = r/2^x = r/2^z = M$ is odd, which implies that $j - i$ is odd.

- **Claim I:** Suppose that $0 \leq i < j$. For a fixed integer $r = 2^z M$ such that M is odd and $1 \leq r < 2^a$, exactly one of the sets $\mathcal{E}_{i,j}^z$ of orbits of θ on $\mathbb{Z}_{2^a}^{(r)}$ has odd cardinality.

- **Proof of Claim I:** First, note that the set S_0^z contains exactly 2^{a-z-1} odd elements and 2^{a-z-1} even elements. Thus for $j > 2^{a-z-1}$, we have $\mathcal{E}_{i,j}^z = \emptyset$, which has even cardinality. Hence we need only consider the case where $0 \leq i < j \leq 2^{a-z-1}$.

  We will count the number of orbits in $\mathcal{E}_{i,j}^z$ where i and j are nonnegative integers such that $i < r/2^{z+1}$ and $i + j = r/2^z = M$. The number of ways to choose i

even elements and j odd elements from the set S_0^z is $\binom{2^{a-z-1}}{i}\binom{2^{a-z-1}}{j}$, which is also equal to the number of ways to choose i odd elements and j even elements from this set. Hence the number of edges which lie in $\mathcal{U}(\mathcal{E}_{i,j}^z)$ is $2\binom{2^{a-z-1}}{i}\binom{2^{a-z-1}}{j}$. Since each orbit of $\mathcal{E}_{i,j}^z$ has length 2^{a-z}, the number of orbits in $\mathcal{E}_{i,j}^z$ is

$$\left|\mathcal{E}_{i,j}^z\right| = \frac{1}{2^{a-z-1}}\binom{2^{a-z-1}}{i}\binom{2^{a-z-1}}{j}. \tag{3.2.1}$$

Case 1: $z = a-1$. In this case we have $r = 2^{a-1}M < 2^a$ for odd M, which implies that $M = 1$ and $r = 2^{a-1}$. Since $i + j = r/2^z = 2^{a-1}/2^{a-1} = 1$ and $i < j$, we must have $i = 0$ and $j = 1$, and so

$$\left|\mathcal{E}_{i,j}^z\right| = \left|\mathcal{E}_{0,1}^{a-1}\right| = \frac{1}{2^0}\binom{2^0}{0}\binom{2^0}{1} = 1,$$

which is odd.

Case 2: $z < a-1$. In this case, since $i + j = M$ is odd, and the cardinality in (3.2.1) is an integer, Lemma A.0.15 implies that $\left|\mathcal{E}_{i,j}^z\right|$ is odd if and only if $i \in \{0, 2^{a-z-1}\}$ or $j \in \{0, 2^{a-z-1}\}$. We will show that exactly one of these situations occurs for $i < j$.

Since $0 \le i < j \le 2^{a-z-1}$, it follows that $j \ne 0$ and $i \ne 2^{a-z-1}$. Hence we need only check that exactly one of the conditions $i = 0$ and $j = 2^{a-z-1}$ hold. Since $z < a-1$, we must have $r \ne 2^{a-1}$. Suppose $r < 2^{a-1}$. Then if $j = 2^{a-z-1}$, we have $i = r/2^z - j = r/2^z - 2^{a-z-1} < 2^{a-z-1} - 2^{a-z-1} = 0$, contradicting the assumption that $i \ge 0$. However, there are edges such that $i = 0$ and $j = r/2^z < 2^{a-1}/2^z = 2^{a-z-1}$. On the other hand, if $r > 2^{a-1}$, then if $i = 0$, we have $j = r/2^z > 2^{a-1}/2^z = 2^{a-z-1}$, and so $j > 2^{a-z-1}$, giving a contradiction. However, there are edges such that $j = 2^{a-z-1}$, for in this case $i = r/2^z - j = r/2^z - 2^{a-z-1} > 2^{a-1}/2^z - 2^{a-z-1} = 0$, so $0 < i < 2^{a-z-1}$.

We have shown that $j \ne 0$, $i \ne 2^{a-z-1}$, and that if $r < 2^{a-1}$ then $j \ne 2^{a-z-1}$

but there exist orbits with $i = 0$ and $j = r/2^z < 2^{a-z-1}$, and that if $r > 2^{a-1}$ then $i \neq 0$ but there exist orbits with $j = 2^{a-z-1}$ and $i = r/2^z - j > 0$.

Thus exactly one of the two situations $i = 0$ and $j = 2^{a-z-1}$ occurs for $i < j$, and neither of the two situations $i = 2^{a-z-1}$ and $j = 0$ can occur. Thus if $z < a-1$, exactly one of $\left|\mathcal{E}^z_{0,r/2^z}\right|$ and $\left|\mathcal{E}^z_{(r/2^z-2^{a-z-1}),2^{a-z-1}}\right|$ is odd, and $\left|\mathcal{E}^z_{i,j}\right|$ is even for all other feasible pairs i, j. This completes the proof of Claim I.

Claim I and the comments preceding it imply that

$$setval_{\hat{\mathcal{E}}_r}(0) - setval_{\hat{\mathcal{E}}_r^C}(0) \text{ is odd}$$

for all integers r such that $1 \leq r < 2^a$. Now fix an integer k such that $2 \leq k < 2^a$. Then $1 \leq k-1 < 2^a$, and so

$$\begin{aligned} & setval_{\hat{\mathcal{E}}_k \cup \hat{\mathcal{E}}_{k-1}}(0) - setval_{\hat{\mathcal{E}}_k^C \cup \hat{\mathcal{E}}_{k-1}^C}(0) \\ &= \Big(setval_{\hat{\mathcal{E}}_k}(0) - setval_{\hat{\mathcal{E}}_k^C}(0)\Big) + \Big(setval_{\hat{\mathcal{E}}_{k-1}}(0) - setval_{\hat{\mathcal{E}}_{k-1}^C}(0)\Big) \end{aligned} \tag{3.2.2}$$

is even.

Now we will find subsets $\mathcal{E}'_{k-1} \subseteq \mathbb{Z}_{2^a}^{(k-1)}$ and $\mathcal{E}'_k \subseteq \mathbb{Z}_{2^a}^{(k)}$ which are related to $\hat{\mathcal{E}}_{k-1}$ and $\hat{\mathcal{E}}_k$, but for which the even quantity in (3.2.2) is bounded. Now for each $r \in \{k-1, k\}$, if $r = 2^z M$, then for all integers x, i and j such that $0 \leq x \leq z$, $0 \leq i \leq r/2^{x+1}$, and $j = r/2^x - i$, we define $\lambda^r(i,j,x)$ as

$$\lambda^r(i,j,x) = setval_{\mathcal{U}(\mathcal{E}^x_{i,j} \cap \hat{\mathcal{E}}_r)}(0) - setval_{\mathcal{U}(\mathcal{E}^x_{i,j} \cap \hat{\mathcal{E}}_r^C)}(0) = j - i.$$

Note that $0 \leq j - i \leq r$. Thus $setval_{\hat{\mathcal{E}}_r}(0) - setval_{\hat{\mathcal{E}}_r^C}(0)$ is equal to the sum of a set A_r of nonnegative integers for

$$A_r = \{\lambda^r(i,j,x) : 0 \leq x \leq z, 0 \leq i \leq r/2^{x+1}, j = r/2^x - i\},$$

and each $\lambda \in A_r$ satisfies $0 \leq \lambda \leq r$. Hence Lemma A.0.16 implies that there is a function $v : A_r \to \{-1, 1\}$ such that $0 \leq \sum_{\lambda \in A_r} \lambda v(\lambda) \leq r$.

Now form a subset $\mathcal{E}'_r$ of $\mathbb{Z}_{2^a}^{(r)}$ from $\hat{\mathcal{E}}_r$ by swapping red edges for blue edges, and vice versa, in $\mathcal{E}^x_{i,j} \cap \hat{\mathcal{E}}_r$ whenever $v(\lambda^r(i,j,x)) = -1$. Then $setval_{\mathcal{E}'_r}(0) - setval_{(\mathcal{E}'_r)^C}(0)$ has the same parity as $setval_{\hat{\mathcal{E}}_r}(0) - setval_{\hat{\mathcal{E}}_r^C}(0)$. Moreover,

$$setval_{\mathcal{E}'_r}(0) - setval_{(\mathcal{E}'_r)^C}(0) = \sum_{\lambda \in A_r} \lambda v(\lambda)$$

and so

$$0 \le setval_{\mathcal{E}'_r}(0) - setval_{(\mathcal{E}'_r)^C}(0) \le r.$$

Thus

$$\begin{aligned} & setval_{\mathcal{E}'_k \cup \mathcal{E}'_{k-1}}(0) - setval_{(\mathcal{E}'_k)^C \cup (\mathcal{E}'_{k-1})^C}(0) \\ &= \Big(setval_{\mathcal{E}'_k}(0) - setval_{(\mathcal{E}'_k)^C}(0)\Big) + \Big(setval_{\mathcal{E}'_{k-1}}(0) - setval_{(\mathcal{E}'_{k-1})^C}(0)\Big) \end{aligned} \tag{3.2.3}$$

is equal to a nonnegative even number 2μ such that $2\mu \le 2k-1$. But 2μ is even and $2k-1$ is odd, so we must have $2\mu \le 2k-2$, which implies that $0 \le \mu \le k-1$.

Case 1: $2 \le k \le 2^{a-1}$. Since k or $k-1$ is even, it follows that $r - \mu$ is even for some $r \in \{k, k-1\}$. Fix this r. Then the system

$$-i + j = \mu$$

$$i + j = r$$

has an integer solution $i = (r-\mu)/2$, $j = (r+\mu)/2$. Also, since $0 \le \mu \le r$, we are guaranteed that $0 \le i, j \le r$, and since $r \le k \le 2^{a-1}$ we also have $0 \le i, j \le 2^{a-1}$. For this r there is an orbit $\mathcal{O} \in \mathcal{E}^0_{i,j}$ of θ on $\mathbb{Z}_{2^a}^{(r)}$ of full length 2^{a-0} such that $\mathcal{E}'_r$ contains the red edges of $\mathcal{O}$, and

$$setval_{\mathcal{O}_{blue}}(0) - setval_{\mathcal{O}_{red}}(0) = j - i = \mu.$$

Let $\mathcal{E}_{k-1} \cup \mathcal{E}_k$ be the set of edges in $\mathbb{Z}_{2^a}^{(k-1)} \cup \mathbb{Z}_{2^a}^{(k)}$ obtained from $\mathcal{E}'_{k-1} \cup \mathcal{E}'_k$ by swapping red edges for blue edges in the orbit $\mathcal{O}$. Then (3.2.3) implies that

$$setval_{\mathcal{E}_k \cup \mathcal{E}_{k-1}}(0) - setval_{\mathcal{E}_k^C \cup \mathcal{E}_{k-1}^C}(0)$$

$$= \left(setval_{\mathcal{E}'_k \cup \mathcal{E}'_{k-1}}(0) - setval_{(\mathcal{E}'_k)^C \cup (\mathcal{E}'_{k-1})^C}(0) \right) - 2\mu$$
$$= 2\mu - 2\mu = 0. \tag{3.2.4}$$

Finally, define $\mathcal{X}_k$ to be the hypergraph with vertex set $\mathcal{V} = \mathbb{Z}_{2^a} \cup \{\infty\}$ and edge set $\mathcal{E} = \mathcal{E}_k \cup \{E \cup \{\infty\} : E \in \mathcal{E}_{k-1}\}$. Since θ maps red edges onto blue edges within each orbit, and vice versa, it follows that $\theta \in \text{Ant}(\mathcal{X}_k)$, and so $\mathcal{X}_k$ is self-complementary. Moreover, (3.2.4) implies that $val_{\mathcal{X}_k}(0) = val_{\mathcal{X}_k^C}(0)$, and so Lemma 3.2.2(2) guarantees that $\mathcal{X}_k$ is 1-subset-regular.

Case 2: $2^{a-1} < k < 2^a$. In this case let $\hat{k} = 2^a - (k-1)$. Then $2 \le \hat{k} \le 2^{a-1}$ and so by *Case 1* there exists a 1-subset-regular self-complementary $\hat{k}$-hypergraph $\mathcal{X}_{\hat{k}}$ on $\mathcal{V}$ with antimorphism θ. Let $\mathcal{F}_{\hat{k}}$ denote the set of edges of $\mathcal{X}_{\hat{k}}$ which do not contain ∞, and let

$$\mathcal{F}_{\hat{k}-1} = \{E \setminus \{\infty\} : E \in \mathcal{E}\left(\mathcal{X}_{\hat{k}}\right), \infty \in E\}.$$

Since $\mathcal{X}_{\hat{k}}$ is 1-subset-regular and self-complementary, it follows that

$$setval_{\mathcal{F}_{\hat{k}} \cup \mathcal{F}_{\hat{k}-1}}(0) = setval_{\mathcal{F}_{\hat{k}}^C \cup \mathcal{F}_{\hat{k}-1}^C}(0). \tag{3.2.5}$$

Let

$$\mathcal{E}_{k-1} = \{\mathbb{Z}_{2^a} \setminus E : E \in \mathcal{F}_{\hat{k}}\}$$

and

$$\mathcal{E}_k = \{\mathbb{Z}_{2^a} \setminus E : E \in \mathcal{F}_{\hat{k}-1}\}.$$

Then $\mathcal{E}_{k-1} \subset \mathbb{Z}_{2^a}^{(k-1)}$ and $\mathcal{E}_k \subset \mathbb{Z}_{2^a}^{(k)}$. Moreover, (3.2.5) implies that

$$setval_{\mathcal{E}_k \cup \mathcal{E}_{k-1}}(0) = setval_{\mathcal{E}_k^C \cup \mathcal{E}_{k-1}^C}(0). \tag{3.2.6}$$

Now define $\mathcal{X}_k$ to be the hypergraph with vertex set $\mathcal{V} = \mathbb{Z}_{2^a} \cup \{\infty\}$ and edge set $\mathcal{E} = \mathcal{E}_k \cup \{E \cup \{\infty\} : E \in \mathcal{E}_{k-1}\}$. Then $\mathcal{X}_k$ is a k-hypergraph on $\mathcal{V}$, and since

$\theta \in \text{Ant}(\mathcal{X}_{\hat{k}})$ it follows that $\theta \in \text{Ant}(\mathcal{X}_k)$, and so $\mathcal{X}_k$ is self-complementary. Moreover, (3.2.6) implies that $val_{\mathcal{X}_k}(0) = val_{\mathcal{X}_k^C}(0)$, and so Lemma 3.2.2(2) guarantees that $\mathcal{X}_k$ is 1-subset-regular, as required. ∎

We are on our way to proving the sufficiency of condition (3.2.7) in the main result of this section, Theorem 3.2.6. In the next lemma, we state and prove the base case for the inductive proof of this sufficiency, which is given in Lemma 3.2.5.

Lemma 3.2.4 *Let a, k, and m be positive integers such that $a \geq 2$ and $k_{[2^a]} \geq 2$. Let $\mathcal{R} = \mathbb{Z}_{m2^a}$, and let $\mathcal{V} = \{\infty\} \cup \mathcal{R}$. There exists a 1-subset-regular self-complementary k-hypergraph on $\mathcal{V}$ with antimorphism*

$$\theta = (\infty) \prod_{j=0}^{m-1} (j2^a, j2^a + 1, \ldots, (j+1)2^a - 1).$$

Proof: We will construct a 1-subset-regular self-complementary k-hypergraph $\mathcal{Y}_k$ on $\mathcal{V}$ with antimorphism θ.

For each $j \in \mathbb{Z}_m$, let

$$\mathcal{R}_j = \{j2^a, j2^a + 1, \ldots, (j+1)2^a - 1\},$$

and let

$$\theta_j = (\infty)(j2^a, j2^a + 1, \ldots, (j+1)2^a - 1) \in Sym(\mathcal{R}_j \cup \{\infty\}).$$

By Lemma 3.2.3, there exists a 1-subset-regular self-complementary r-hypergraph Δ_r^j on $\{\infty\} \cup \mathcal{R}_j$, with antimorphism θ_j, for $r \in \{2, 3, \ldots, 2^a - 1\}$.

For each $E \in \mathcal{V}^{(k)}$, let $C_1(E) = \{j \in \mathbb{Z}_m : 2 \leq |(\{\infty\} \cup \mathcal{R}_j) \cap E| \leq 2^a - 1\}$. If $C_1(E) \neq \emptyset$, set $j_1(E) = \min\{j : j \in C_1(E)\}$. If $E \in \mathcal{V}^{(k)}$ and $C_1(E) = \emptyset$, then $|(\{\infty\} \cup \mathcal{R}_j) \cap E| \leq 1$ or $|(\{\infty\} \cup \mathcal{R}_j) \cap E| \geq 2^a$ for all $j \in \mathbb{Z}_m$. Since $2 \leq k_{[2^a]} < 2^a$, this implies that one of the following conditions hold when $C_1(E) = \emptyset$:

- $\infty \notin E$, all cycles of θ contain $0, 1$, or 2^a elements of E, and at least two nontrivial cycles of θ contain exactly one element of E.

- $\infty \in E$, all nontrivial cycles of θ contain at least $2^a - 1$ elements of E, and at least two nontrivial cycles of θ contain exactly $2^a - 1$ elements of E.

For each $E \in \mathcal{V}^{(k)}$ with $C_1(E) = \emptyset$, define $C_2(E) = \{j \in \mathbb{Z}_m : |(E \cap \mathcal{R}_j)| \in \{1, 2^a - 1\}\}$. Then $|C_2(E)| \geq 2$. Let $i_1(E)$ and $i_2(E)$ be the two smallest elements of $C_2(E)$.

Now define $\mathcal{Y}_k$ to be the k-hypergraph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$ such that an element $E \in \mathcal{V}^{(k)}$ is in $\mathcal{E}$ if and only if one of the following conditions hold for $j_1 = j_1(E)$, $i_1 = i_1(E)$, and $i_2 = i_2(E)$.

(i) $C_1(E) \neq \emptyset$, $|E \cap (\{\infty\} \cup \mathcal{R}_{j_1})| = r$, and $E \cap (\{\infty\} \cup \mathcal{R}_{j_1}) \in \mathcal{E}\,(\Delta_r^{j_1})$.

(ii) $C_1(E) = \emptyset$, $\infty \notin E$, $E \cap \mathcal{R}_{i_1} = \{x\}$, $E \cap \mathcal{R}_{i_2} = \{y\}$, and $(x + y)_{[4]} \in \{1, 2\}$.

(iii) $C_1(E) = \emptyset$, $\infty \in E$, $\mathcal{R}_{i_1} \setminus E = \{x\}$, $\mathcal{R}_{i_2} \setminus E = \{y\}$, and $(x + y)_{[4]} \in \{1, 2\}$.

We will prove that $\mathcal{Y}_k$ is 1-subset-regular and self-complementary with antimorphism θ.

First we will show that $\mathcal{Y}_k$ is self-complementary. Note that $\mathcal{E}^C = \mathcal{V}^{(k)} \setminus \mathcal{E}$ is the set of elements E of $\mathcal{V}^{(k)}$ for which one of the following conditions hold. (Again, $j_1 = j_1(E)$, $i_1 = i_1(E)$, and $i_2 = i_2(E)$.)

(i)$'$ $C_1(E) \neq \emptyset$, $|E \cap (\{\infty\} \cup \mathcal{R}_{j_1})| = r$, and $E \cap (\{\infty\} \cup \mathcal{R}_{j_1}) \notin \mathcal{E}\,(\Delta_r^{j_1})$.

(ii)$'$ $C_1(E) = \emptyset$, $\infty \notin E$, $E \cap \mathcal{R}_{i_1} = \{x\}$, $E \cap \mathcal{R}_{i_2} = \{y\}$, $(x + y)_{[4]} \in \{0, 3\}$.

(iii)$'$ $C_1(E) = \emptyset$, $\infty \in E$, $\mathcal{R}_{i_1} \setminus E = \{x\}$, $\mathcal{R}_{i_2} \setminus E = \{y\}$, and $(x + y)_{[4]} \in \{0, 3\}$.

Observe that $\theta \mid_{\{\infty\} \cup \mathcal{R}_{j_1}} = \theta_{j_1} \in \mathrm{Ant}(\Delta_r^{j_1})$. Hence an element $E \in \mathcal{V}^{(k)}$ satisfies condition (i) if and only if E^θ satisfies condition (i)$'$. Also, for $x \in R_{i_1}$, $y \in R_{i_2}$, and $a \geq 2$, we have $(x^\theta + y^\theta)_{[4]} = ((x+1)_{[2^a]} + (y+1)_{[2^a]})_{[4]} = (x + y + 2)_{[4]}$, so θ maps elements x and y with $(x, y) \in \mathcal{R}_{i_1} \times \mathcal{R}_{i_2}$ and $(x + y)_{[4]} \in \{1, 2\}$ to elements x^θ and y^θ with $(x^\theta, y^\theta) \in \mathcal{R}_{i_1} \times \mathcal{R}_{i_2}$ and $(x^\theta + y^\theta)_{[4]} \in \{0, 3\}$, and vice versa. It follows that an element $E \in \mathcal{V}^{(k)}$ satisfies condition (ii) if and only if E^θ satisfies condition (ii)$'$, and E satisfies condition (iii) if and only if E^θ satisfies condition (iii)$'$. Hence $E \in \mathcal{E}$ if and only if $E^\theta \in \mathcal{E}^C$. Thus $\theta \in \mathrm{Ant}(\mathcal{Y}_k)$ and $\mathcal{Y}_k$ is self-complementary.

Next we show that $\mathcal{Y}_k$ is 1-subset-regular, which by Lemma 3.2.2(1) is true if and only if $val_{\mathcal{Y}_k}(v) = val_{\mathcal{Y}_k^C}(v)$ for all $v \in \mathcal{V}$. Since $\theta \in \mathrm{Ant}(\mathcal{Y}_k)$ and θ fixes ∞, we certainly have $val_{\mathcal{Y}_k}(\infty) = val_{\mathcal{Y}_k^C}(\infty)$. It remains to show that $val_{\mathcal{Y}_k}(v) = val_{\mathcal{Y}_k^C}(v)$ for all $v \in \mathcal{R}$.

Let $j' \in \mathbb{Z}_m$ and suppose that $v \in \mathcal{R}_{j'}$. Let $\mathcal{O}$ be an orbit of θ on $\mathcal{V}^{(k)}$ which contains edges containing v. Let $E \in \mathcal{O}$, and set $C_1(\mathcal{O}) = C_1(E)$, and if $C_1(E) \neq \emptyset$, set $j_1(\mathcal{O}) = j_1(E)$. Note that $C_1(E)$ is constant over all $E \in \mathcal{O}$, and so $C_1(\mathcal{O})$ is independent of our choice of $E \in \mathcal{O}$, and so is $j_1(\mathcal{O})$, if it exists. If $C_1(\mathcal{O}) = \emptyset$, set $C_2(\mathcal{O}) = C_2(E)$, and set $i_1(\mathcal{O}) = i_1(E)$ and $i_2(\mathcal{O}) = i_2(E)$. If $C_1(\mathcal{O}) = \emptyset$, then $C_2(\mathcal{O})$ is constant over all $E \in \mathcal{O}$, and so $C_2(\mathcal{O})$, $i_1(\mathcal{O})$, and $i_2(\mathcal{O})$ are also independent of our choice of E. Now $\mathcal{O}$ is one of four types:

- TYPE 1: $C_1(\mathcal{O}) \neq \emptyset$ and $j' \neq j_1(\mathcal{O})$.
- TYPE 2: $C_1(\mathcal{O}) \neq \emptyset$ and $j' = j_1(\mathcal{O})$.
- TYPE 3: $C_1(\mathcal{O}) = \emptyset$ and $j' \notin \{i_1(\mathcal{O}), i_2(\mathcal{O})\}$.
- TYPE 4: $C_1(\mathcal{O}) = \emptyset$ and $j' \in \{i_1(\mathcal{O}), i_2(\mathcal{O})\}$.

For each $i \in \{1,2,3,4\}$, let $\mathcal{P}_i$ be the set of orbits of θ on $\mathcal{V}^{(k)}$ of TYPE i which contain edges containing v. We will show that

$$setval_{\mathcal{U}(\mathcal{P}_i)\cap\mathcal{E}}(v) = setval_{\mathcal{U}(\mathcal{P}_i)\cap\mathcal{E}^C}(v)$$

for all $i \in \{1,2,3,4\}$. For each i, let $(\mathcal{U}(\mathcal{P}_i) \cap \mathcal{E})_v = \{E \in \mathcal{U}(\mathcal{P}_i) \cap \mathcal{E} : v \in E\}$, and let $(\mathcal{U}(P_i) \cap \mathcal{E}^C)_v = \{E \in \mathcal{U}(\mathcal{P}_i) \cap \mathcal{E}^C : v \in E\}$.

First consider the orbits of $\mathcal{P}_1$. Define the mapping $\beta_1 : (\mathcal{U}(\mathcal{P}_1)\cap\mathcal{E})_v \to (\mathcal{U}(\mathcal{P}_1)\cap \mathcal{E}^C)_v$ by

$$E^{\beta_1} = (\mathcal{R}_{j_1} \cap E^\theta) \cup (E \setminus \mathcal{R}_{j_1}),$$

for all $E \in (\mathcal{U}(\mathcal{P}_1) \cap \mathcal{E})_v$, where $j_1 = j_1(\mathcal{O})$ for the orbit $\mathcal{O}$ of θ on $\mathcal{V}^{(k)}$ containing E.

Since $j' \neq j_1$ for all orbits $\mathcal{O} \in \mathcal{P}_1$, and since $v \in \mathcal{R}_{j'}$, it follows that for all $E \in (\mathcal{U}(\mathcal{P}_1) \cap \mathcal{E})_v$ we have $v \in E \setminus \mathcal{R}_{j_1}$. Hence β_1 maps edges of $(\mathcal{U}(\mathcal{P}_1) \cap \mathcal{E})_v$ to edges of $(\mathcal{U}(\mathcal{P}_1) \cap \mathcal{E}^C)_v$. Moreover, one can verify that β_1 is invertible, with inverse β_1^{-1} defined by

$$E^{\beta_1^{-1}} = \left(\mathcal{R}_{j_1} \cap E^{\theta^{-1}}\right) \cup (E \setminus \mathcal{R}_{j_1}),$$

for all $E \in (\mathcal{U}(\mathcal{P}_1) \cap \mathcal{E}^C)_v$, where $j_1 = j_1(\mathcal{O})$ for the orbit $\mathcal{O}$ of θ on $\mathcal{V}^{(k)}$ containing E. We conclude that $|(\mathcal{U}(\mathcal{P}_1) \cap \mathcal{E})_v| = |(\mathcal{U}(\mathcal{P}_1) \cap \mathcal{E}^C)_v|$, and hence

$$setval_{\mathcal{U}(\mathcal{P}_1)\cap\mathcal{E}}(v) = setval_{\mathcal{U}(\mathcal{P}_1)\cap\mathcal{E}^C}(v).$$

Now consider the orbits of $\mathcal{P}_2$. Every orbit $\mathcal{O}$ of $\mathcal{P}_2$ satisfies $j_1(\mathcal{O}) = j'$, and so $E \cap (\{\infty\} \cup \mathcal{R}_{j_1}) \in \mathcal{E}(\Delta_r^{j_1})$, where $r = |E \cap (\{\infty\} \cup \mathcal{R}_{j_1})|$, for all $E \in \mathcal{O} \cap \mathcal{E}$. Observe that since $\Delta_r^{j_1}$ is 1-subset-regular and self-complementary for all r, by Lemma 3.2.2(1) we have

$$val_{\Delta_r^{j_1}}(v) = val_{(\Delta_r^{j_1})^C}(v).$$

This implies that there is a bijection δ between the set of edges of $\Delta_r^{j_1}$ containing v and the set of edges of $(\Delta_r^{j_1})^C$ containing v. Now define the mapping $\beta_2 : (\mathcal{U}(\mathcal{P}_2) \cap \mathcal{E})_v \to (\mathcal{U}(\mathcal{P}_2) \cap \mathcal{E}^C)_v$ by

$$E^{\beta_2} = (E \cap (\{\infty\} \cup \mathcal{R}_{j_1}))^{\delta} \cup (E \setminus (\{\infty\} \cup \mathcal{R}_{j_1})),$$

for all $E \in (\mathcal{U}(\mathcal{P}_2) \cap \mathcal{E})_v$, where $j_1 = j_1(\mathcal{O})$ for the orbit $\mathcal{O}$ of θ on $\mathcal{V}^{(k)}$ containing E.

Since $j' = j_1$ for all orbits $\mathcal{O} \in \mathcal{P}_2$, and $v \in \mathcal{R}_{j_1}$, the definition of δ guarantees that $v \in E \cap (\{\infty\} \cup \mathcal{R}_{j_1})$ if and only if $v \in (E \cap (\{\infty\} \cup \mathcal{R}_{j_1}))^{\delta}$. Also, condition (i) guarantees that $E \in \mathcal{U}(\mathcal{P}_2) \cap \mathcal{E}$ if and only if $E^{\beta_2} \in \mathcal{U}(\mathcal{P}_2) \cap \mathcal{E}^C$. Hence β_2 maps edges of $(\mathcal{U}(\mathcal{P}_2) \cap \mathcal{E})_v$ to edges of $(\mathcal{U}(\mathcal{P}_2) \cap \mathcal{E}^C)_v$. Moreover, one can verify that β_2 is invertible, with inverse β_2^{-1} defined by

$$E^{\beta_2^{-1}} = (E \cap (\{\infty\} \cup \mathcal{R}_{j_1}))^{\delta^{-1}} \cup (E \setminus (\{\infty\} \cup \mathcal{R}_{j_1})),$$

for all $E \in (\mathcal{U}(\mathcal{P}_2) \cap \mathcal{E}^C)_v$, where $j_1 = j_1(\mathcal{O})$ for the orbit $\mathcal{O}$ of θ on $\mathcal{V}^{(k)}$ containing E. We conclude that $|(\mathcal{U}(\mathcal{P}_2) \cap \mathcal{E})_v| = |(\mathcal{U}(\mathcal{P}_2) \cap \mathcal{E}^C)_v|$, and hence

$$setval_{\mathcal{U}(\mathcal{P}_2)\cap\mathcal{E}}(v) = setval_{\mathcal{U}(\mathcal{P}_2)\cap\mathcal{E}^C}(v).$$

Now consider the orbits of $\mathcal{P}_3$. Define the mapping $\beta_3 : (\mathcal{U}(\mathcal{P}_3) \cap \mathcal{E})_v \to (\mathcal{U}(\mathcal{P}_3) \cap \mathcal{E}^C)_v$ by

$$E^{\beta_3} = ((\mathcal{R}_{i_1} \cup \mathcal{R}_{i_2}) \cap E^{\theta}) \cup (E \setminus (\mathcal{R}_{i_1} \cup \mathcal{R}_{i_2})),$$

for all $E \in (\mathcal{U}(\mathcal{P}_3) \cap \mathcal{E})_v$, where $i_1 = i_1(\mathcal{O})$ and $i_2 = i_2(\mathcal{O})$ for the orbit $\mathcal{O}$ of θ on $\mathcal{V}^{(k)}$ containing E.

Since $j' \notin \{i_1, i_2\}$ for all orbits $\mathcal{O} \in \mathcal{P}_3$, and $v \in \mathcal{R}_{j'}$, for all $E \in (\mathcal{U}(\mathcal{P}_3) \cap \mathcal{E})_v$ we have $v \in E \setminus (\mathcal{R}_{i_1} \cup \mathcal{R}_{i_2})$. Hence β_3 maps edges of $(\mathcal{U}(\mathcal{P}_3) \cap \mathcal{E})_v$ to edges of $(\mathcal{U}(\mathcal{P}_3) \cap \mathcal{E}^C)_v$. Moreover, one can verify that β_3 is invertible, with inverse β_3^{-1} defined by

$$E^{\beta_3^{-1}} = \left((\mathcal{R}_{i_1} \cup \mathcal{R}_{i_2}) \cap E^{\theta^{-1}}\right) \cup (E \setminus (\mathcal{R}_{i_1} \cup \mathcal{R}_{i_2})),$$

for all $E \in (\mathcal{U}(\mathcal{P}_3) \cap \mathcal{E}^C)_v$, where $i_1 = i_1(\mathcal{O})$ and $i_2 = i_2(\mathcal{O})$ for the orbit $\mathcal{O}$ of θ on $\mathcal{V}^{(k)}$ containing E. We conclude that $|(\mathcal{U}(\mathcal{P}_3) \cap \mathcal{E})_v| = |(\mathcal{U}(\mathcal{P}_3) \cap \mathcal{E}^C)_v|$, and hence

$$setval_{\mathcal{U}(\mathcal{P}_3)\cap\mathcal{E}}(v) = setval_{\mathcal{U}(\mathcal{P}_3)\cap\mathcal{E}^C}(v).$$

Finally, consider the orbits of $\mathcal{P}_4$. Every orbit $\mathcal{O}$ of $\mathcal{P}_4$ satisfies $j' \in \{i_1, i_2\}$. Since $v \in \mathcal{R}_{j'}$, we must have $v \in \mathcal{R}_{i_1} \cup \mathcal{R}_{i_2}$. Assume, without loss of generality, that $v \in \mathcal{R}_{i_1}$. Define the mapping $\beta_4 : (\mathcal{U}(\mathcal{P}_4) \cap \mathcal{E})_v \to (\mathcal{U}(\mathcal{P}_4) \cap \mathcal{E}^C)_v$ by

$$E^{\beta_4} = \left(\mathcal{R}_{i_2} \cap E^{\theta^2}\right) \cup (E \setminus \mathcal{R}_{i_2}),$$

for all $E \in (\mathcal{U}(\mathcal{P}_4) \cap \mathcal{E})_v$, where in each case $i_1 = i_1(\mathcal{O})$ and $i_2 = i_2(\mathcal{O})$ for the orbit $\mathcal{O}$ of θ on $\mathcal{V}^{(k)}$ containing E. Now since $v \in \mathcal{R}_{i_1}$, it follows that $v \in E \setminus \mathcal{R}_{i_2}$ for all E in $(\mathcal{U}(\mathcal{P}_4) \cap \mathcal{E})_v$. Now observe that if $E \in (\mathcal{U}(\mathcal{P}_4) \cap \mathcal{E})_v$, then either $|E \cap \mathcal{R}_{i_1}| = |E \cap \mathcal{R}_{i_2}| = 1$ or $|\mathcal{R}_{i_1} \setminus E| = |\mathcal{R}_{i_2} \setminus E| = 1$. In the former case, we must have $E \cap \mathcal{R}_{i_1} = \{v\}$ and $E \cap \mathcal{R}_{i_2} = \{w\}$, for some $w \in \mathcal{R}_{i_2}$ such that $(v+w)_{[4]} \in \{1,2\}$, which implies that $E^{\beta_4} \cap \mathcal{R}_{i_1} = \{v\}$, $E^{\beta_4} \cap \mathcal{R}_{i_2} = \{(w+2)_{[2^a]}\}$, and $(v + (w+2)_{[2^a]})_{[4]} = (v+w+2)_{[4]} \in \{0,3\}$, since $a \geq 2$. In the latter case, we must have $\mathcal{R}_{i_1} \setminus E = \{x\}$ and $\mathcal{R}_{i_2} \setminus E = \{y\}$, for some $x \in \mathcal{R}_{i_1}$ and $y \in \mathcal{R}_{i_2}$ such that $x \neq v$ and $(x+y)_{[4]} \in \{1,2\}$, which implies that $\mathcal{R}_{i_1} \setminus E^{\beta_4} = \{x\}$, $\mathcal{R}_{i_2} \setminus E^{\beta_4} = \{(y+2)_{[2^a]}\}$, and $(x + (y+2)_{[2^a]})_{[4]} = (x+y+2)_{[4]} \in \{0,3\}$, since $a \geq 2$. Hence conditions (ii) and (iii) guarantee that β_4 maps edges of $(\mathcal{U}(\mathcal{P}_4) \cap \mathcal{E})_v$ to edges of $(\mathcal{U}(\mathcal{P}_4) \cap \mathcal{E}^C)_v$. Moreover, the permutation β_4 is invertible, with inverse β_4^{-1} defined by

$$E^{\beta_4^{-1}} = \left(\mathcal{R}_{i_2} \cap E^{\theta^{-2}}\right) \cup (E \setminus \mathcal{R}_{i_2}),$$

for all $E \in (\mathcal{U}(\mathcal{P}_4) \cap \mathcal{E}^C)_v$, where $i_1 = i_1(\mathcal{O})$ and $i_2 = i_2(\mathcal{O})$ for the orbit $\mathcal{O}$ of θ on $\mathcal{V}^{(k)}$ containing E. We conclude that $|(\mathcal{U}(\mathcal{P}_4) \cap \mathcal{E})_v| = |(\mathcal{U}(\mathcal{P}_4) \cap \mathcal{E}^C)_v|$, and hence

$$setval_{\mathcal{U}(\mathcal{P}_4)\cap\mathcal{E}}(v) = setval_{\mathcal{U}(\mathcal{P}_4)\cap\mathcal{E}^C}(v).$$

Now observe that

$$val_{\mathcal{Y}_k}(v) = \sum_{i=1}^{4} setval_{\mathcal{U}(\mathcal{P}_i)\cap\mathcal{E}}(v) = \sum_{i=1}^{4} setval_{\mathcal{U}(\mathcal{P}_i)\cap\mathcal{E}^C}(v) = val_{\mathcal{Y}_k^C}(v).$$

Since j' was an arbitrary element of $\mathbb{Z}_m$, we conclude that $val_{\mathcal{Y}_k}(v) = val_{\mathcal{Y}_k^C}(v)$ for all $v \in \mathcal{R} = \cup_{j \in \mathbb{Z}_m} \mathcal{R}_j$, and hence for all $v \in \mathcal{V} = \mathcal{R} \cup \{\infty\}$. Thus Lemma 3.2.2(1) implies that $\mathcal{Y}_k$ is 1-subset-regular. ∎

It should be noted that Lemma 3.2.4 was proved previously for the case where $a = 2$. Rao handled the case where $a = 2$ and $k = 2$ in [27], and Potočnik and Šajna handled the case where $a = 2$ and $k = 3$ in [23].

We are ready to prove the sufficiency of condition (3.1.3) in Theorem 3.1.2. Lemma 3.2.5 demonstrates the existence of a 1-subset-regular self-complementary uniform hypergraph of rank k and order n for every pair (n, k) satisfying condition (3.2.7).

Lemma 3.2.5 *Let a, k, m, and s be positive integers such that $a \geq 2$ and $s < k_{[2^a]}$. Let $\mathcal{R} = \mathbb{Z}_{m2^a}$, let $S = \{\infty_1, \infty_2, \ldots, \infty_s\}$ such that $S \cap \mathcal{R} = \emptyset$, and let $\mathcal{V} = S \cup \mathcal{R}$. There exists a 1-subset-regular self-complementary k-hypergraph on $\mathcal{V}$ with antimorphism*

$$\theta = (\infty_1)(\infty_2)\cdots(\infty_s)\prod_{j=0}^{m-1}(j2^a, j2^a+1, \ldots, (j+1)2^a - 1).$$

Proof: Fix positive integers a and m such that $a \geq 2$. We prove that there exists a 1-subset-regular self-complementary k-hypergraph on $\mathcal{V}$ with antimorphism θ for all positive integers k and s such that $1 \leq s < k_{[2^a]}$. The proof is by induction on s.

Base Step: $s = 1$. In this case, since $s < k_{[2^a]}$, we have $k_{[2^a]} \geq 2$, and so the existence of a 1-subset-regular self-complementary k-hypergraph on $\mathcal{V} = \{\infty_1\} \cup \mathbb{Z}_{m2^a}$ with antimorphism

$$\theta = (\infty_1)\prod_{j=0}^{m-1}(j2^a, j2^a+1, \ldots, (j+1)2^a - 1)$$

follows from Lemma 3.2.4. Hence the result holds for $s = 1$.

Induction Step: Suppose $s > 1$, and assume that there exists a 1-subset-regular self-complementary $\hat{k}$-hypergraph $Z_{\hat{k}}$ on

$$\hat{\mathcal{V}} = \{\infty_1, \ldots, \infty_{s-1}\} \cup \mathbb{Z}_{m2^a}$$

with antimorphism

$$\hat{\theta} = (\infty_1)\cdots(\infty_{s-1})\prod_{j=0}^{m-1}(j2^a, j2^a+1, \ldots, (j+1)2^a - 1),$$

for all $\hat{k}$ such that $1 \leq s - 1 < \hat{k}_{[2^a]}$.

Now let k be a positive integer such that $s < k_{[2^a]}$. We will construct a 1-subset-regular self-complementary k-hypergraph on $\mathcal{V}$ with antimorphism θ. Now $1 \leq s-1 <$

$k_{[2^a]}$ and so by the induction hypothesis, there exists a 1-subset-regular self-complementary k-hypergraph Z_k on $\hat{\mathcal{V}}$ with antimorphism $\hat{\theta}$. Moreover, since $s \geq 2$, we have $k_{[2^a]} \geq 3$, and so $(k-1)_{[2^a]} = k_{[2^a]} - 1$. This implies that $1 \leq s-1 < (k-1)_{[2^a]}$, and so by the induction hypothesis, there also exists a 1-subset-regular self-complementary $(k-1)$-hypergraph Z_{k-1} on $\hat{\mathcal{V}}$ with antimorphism $\hat{\theta}$.

Let $\mathcal{Z}_k$ be the k-hypergraph with vertex set $\mathcal{V} = \hat{\mathcal{V}} \cup \{\infty_s\}$ and edge set

$$\mathcal{E} = \mathcal{E}(Z_k) \bigcup \{\{\infty_s\} \cup E : E \in \mathcal{E}(Z_{k-1})\}.$$

Since $\theta \mid_{\hat{\mathcal{V}}} = \hat{\theta} \in \text{Ant}(Z_k) \cap \text{Ant}(Z_{k-1})$, and θ fixes ∞_s, it follows that $E \in \mathcal{E}$ if and only if $E^\theta \in \mathcal{E}^C$. Hence $\theta \in \text{Ant}(\mathcal{Z}_k)$ and $\mathcal{Z}_k$ is self-complementary. Moreover, for all $v \in \hat{\mathcal{V}}$, we have

$$\begin{aligned} val_{\mathcal{Z}_k}(v) &= val_{Z_k}(v) + val_{Z_{k-1}}(v) \\ &= val_{Z_k^C}(v) + val_{Z_{k-1}^C}(v) \\ &= val_{\mathcal{Z}_k^C}(v). \end{aligned}$$

Since the antimorphism θ fixes ∞_s, we also have $val_{\mathcal{Z}_k}(\infty_s) = val_{\mathcal{Z}_k^C}(\infty_s)$, and so $val_{\mathcal{Z}_k}(v) = val_{\mathcal{Z}_k^C}(v)$ for all $v \in \hat{\mathcal{V}} \cup \{\infty\} = \mathcal{V}$. Thus Lemma 3.2.2(1) implies that $\mathcal{Z}_k$ is 1-subset-regular.

Hence by induction on s, there exists a 1-subset-regular self-complementary k-hypergraph on $\mathcal{V}$ with antimorphism θ for every positive integer $s < k_{[2^a]}$. ∎

Theorem 3.2.6 *Let k and n be positive integers such that $1 < k \leq n$. There exists a 1-subset-regular self-complementary k-hypergraph of order n if and only if there exists an integer a such that $\max\{i : 2^i \mid k\} < a \leq \min\{i : 2^i > k\}$ and*

$$n_{[2^a]} \in \{1, 2, \ldots, k_{[2^a]} - 1\}. \tag{3.2.7}$$

Proof: The necessity of condition (3.2.7) follows from Theorem 3.1.2. Suppose that n satisfies condition (3.2.7). Then $n = m2^a + s$ for some positive integers a, m, and s such that $\max\{i : 2^i \mid k\} < a \leq \min\{i : 2^i > k\}$ and $1 \leq s < k_{[2^a]}$. If $a = 1$, then $1 \leq s < k_{[2^a]}$ cannot hold, and so in this case the sufficiency of condition (3.2.7) holds vacuously. On the other hand, if $a \geq 2$, then the existence of a 1-subset-regular self-complementary k-hypergraph of order n follows from Lemma 3.2.5, and so condition (3.2.7) is sufficient in this case also. ∎

Lemma 3.1.3 states that the necessary and sufficient conditions (3.2.7) of Theorem 3.2.6 are equivalent to the necessary conditions (3.1.5) of Theorem 3.1.4 in the case $t = 1$. We obtain the following alternative statement of the necessary and sufficient conditions on the order of a 1-subset-regular self-complementary k-uniform hypergraph in terms of the binary representation of k.

Theorem 3.2.7 *Let k and n be positive integers such that $1 < k \leq n$, and let b be the binary representation of k. There exists a 1-subset-regular self-complementary k-hypergraph of order n if and only if*

$$n_{[2^{\ell+1}]} \in \{1, 2, \ldots, k_{[2^{\ell+1}]} - 1\} \tag{3.2.8}$$

for some $\ell \in supp(b)$.

3.2.3 Open problem

The author proposes the following problem.

Problem 3.2.8 *For given integers $k \geq 2$ and $t \geq 1$, determine the set $\mathcal{L}_{t,k}$ of all integers n for which there exists a t-subset-regular self-complementary k-hypergraph of order n.*

Theorem 3.2.1 gives a solution to problem 3.2.8 in the case where $k \in \{2, 3\}$, and Theorem 3.2.6 solves this problem for all positive integers k in the case where $t = 1$. However, Problem 3.2.8 remains mainly unsolved.

3.2.4 Connections to design theory

Recall that if a t-subset-regular k-uniform hypergraph X of order n is self-complementary, then X and its complement X^C are both t-(n, k, λ) designs with $\lambda = \binom{n-t}{k-t}/2$. Hence the pair $\{X, X^C\}$ is an $LS[2](t, k, n)$ in which the t-designs are isomorphic.

In [12], Hartman considered the problem of halving the complete t-$(n, k, \binom{n-t}{k-t})$ design into two t-$(n, k, \binom{n-t}{k-t}/2)$ designs to form a $LS[2](t, k, n)$, and he conjectured that the basic necessary conditions on the order n given by Lemma 3.1.1 are also sufficient.

Conjecture 3.2.9 [12] *There exists a $LS[2](t, k, n)$ if and only if $\binom{n-i}{k-i}$ is even for $i = 0, 1, \ldots, t$.*

Baranyai [5] proved that Hartman's conjecture is true for $t = 1$. The combined efforts of the authors Ajoodani-Namini, Alltop, Dehon, Hartman, Khosrovshahi, and Teirlinck in the works [5, 2, 3, 4, 1, 8, 12, 16, 34] proved that Hartman's conjecture is true for $t = 2$. However, it is important to note that this result does not provide a solution to Problem 3.2.8 for the case $t = 2$, since it does not guarantee that there exists a $LS[2](2, k, n)$ for every admissible order of Lemma 3.1.1 in which the 2-designs are *isomorphic*. Hence these results on halving the complete designs do not show the existence of 2-subset-regular self-complementary k-hypergraphs of every admissible order. The only previous result that the author has found on halving the complete design into two isomorphic 2-designs is Knor and Potočnik's construction for 2-subset-regular self-complementary 3-hypergraphs of every admissible order [18].

Chapter 4

Transitive self-complementary hypergraphs

4.1 Necessary conditions on order

4.1.1 Previous results: vertex transitivity

In this section, we present the known necessary conditions on the order n of a vertex transitive self-complementary k-hypergraph X in the case that k has the form $k = 2^\ell$ or $k = 2^\ell + 1$ for some positive integer ℓ and $n \equiv 1 \pmod{2^{\ell+1}}$. Such a k-hypergraph X is necessarily 1-subset-regular. Hence Corollary 3.1.5 implies that

$$n_{[2^{\ell+1}]} \in \{1, 2, \ldots, k-1\}.$$

However, the vertex-transitivity of X implies even stronger necessary conditions on its order n in the case $n \equiv 1 \pmod{2^{\ell+1}}$, as the next result due to Potočnik and Šajna [24] shows.

Recall that for a positive integer n and a prime number p, the symbol $n_{(p)}$ denotes the largest integer i such that p^i divides n.

Theorem 4.1.1 **[24]** *Let ℓ be a positive integer, let $k = 2^\ell$ or $k = 2^\ell + 1$, and let $n \equiv 1 \pmod{2^{\ell+1}}$. If there exists a vertex transitive self-complementary k-hypergraph of order n, then*

$$p^{n_{(p)}} \equiv 1 \pmod{2^{\ell+1}} \quad \textit{for every prime } p.$$

Theorem 4.1.1 was proved by Li [20] for $k = 2$ and $n \equiv 1 \pmod 4$ for the special case when n is a product of two distinct primes. Li's proof is based on a classification of vertex transitive graphs of order pq (where $p \neq q$ are primes), which was obtained by Praeger and Xu [26] using the classification of finite simple groups. In 1999, Muzychuk [21] gave an algebraic proof of Theorem 4.1.1 for the case $k = 2$ and $n \equiv 1 \pmod 4$. Consequently, integers n satisfying $p^{n_{(p)}} \equiv 1 \pmod 4$ for all primes p are called *Muzychuk integers*. In 2007, Potočnik and Šajna extended the idea in Muzychuk's proof to prove Theorem 4.1.1.

The following immediate corollary to Theorem 4.1.1 gives necessary conditions on the prime divisors of the order n of a vertex transitive self-complementary k-hypergraph for small values of the rank k when n satisfies the hypotheses of Theorem 4.1.1.

Corollary 4.1.2 *Suppose X is a vertex transitive self-complementary k-hypergraph of order n, and let p^r be the highest power of a prime p that divides n. Then the following conditions hold.*

(a) If $k = 2$, then $p^r \equiv 1 \pmod 4$.

(b) If $k = 3$ and n is odd, then $p^r \equiv 1 \pmod 4$.

(c) If $k = 4$ or 5, and $n \equiv 1 \pmod 8$, then $p^r \equiv 1 \pmod 8$.

4.1.2 New results: t-fold-transitivity

Suppose that $k = 2^\ell$ or $k = 2^\ell + 1$ for a positive integer ℓ, and that $t \geq 1$ is an integer. If X is a t-transitive self-complementary k-hypergraph of order n, then X is necessarily t-subset-regular. Hence Corollary 3.1.5 implies that

$$n_{[2^{\ell+1}]} \in \{t, t+1, \ldots\ldots, k-1\}.$$

However, the t-transitivity of X implies even stronger necessary conditions on its order n in the cases where $n \equiv t \pmod{2^{\ell+1}}$, as the next result shows.

Theorem 4.1.3 *Let ℓ be a positive integer, let $k = 2^\ell$ or $k = 2^\ell + 1$, let t be a positive integer and let $n \equiv t \pmod{2^{\ell+1}}$. If there exists a t-transitive self-complementary k-hypergraph of order n, then*

$$p^{(n-t+1)_{(p)}} \equiv 1 \pmod{2^{\ell+1}} \quad \textit{for every prime } p.$$

Proof: When $t = 1$ the result follows directly from Theorem 4.1.1, so we may assume that $t \geq 2$.

Suppose that $X = (V, E)$ is a t-transitive self-complementary k-hypergraph of order $n \equiv t \pmod{2^{\ell+1}}$. Let $v_1, v_2, \ldots, v_{t-1} \in V$, and let $\theta \in \mathrm{Ant}(X)$. Since X is t-transitive, it is certainly $(t-1)$-transitive, and so there exists $\sigma \in \mathrm{Aut}(X)$ such that $v_i^{\theta\sigma} = (v_i^\theta)^\sigma = v_i$ for all $i \in \{1, 2, \ldots, t-1\}$. Hence $\theta\sigma$ fixes $\{v_1, \ldots, v_{t-1}\}$ pointwise and $\theta\sigma \in \mathrm{Ant}(X)$. That is, there exists an antimorphism $\theta^* = \theta\sigma$ of X which fixes every element in the set $\{v_1, \ldots, v_{t-1}\}$. Also, since X is t-transitive, it follows that $\bigcap_{i=1}^{t-1} \mathrm{Aut}(X)_{v_i}$ acts transitively on $V \setminus \{v_1, v_2, \ldots, v_{t-1}\}$.

For each $i \in \{1, 2, \ldots, t-1\}$, let E_{v_i} denote the set of edges of E containing v_i, and $\overline{E}_{v_i}$ denote the set of edges of $V^{(k)} \setminus E$ containing v_i. Then every permutation in $\bigcap_{i=1}^{t-1} \mathrm{Aut}(X)_{v_i}$ must map edges in $\bigcup_{i=1}^{t-1} E_{v_i}$ onto edges in $\bigcup_{i=1}^{t-1} E_{v_i}$, and the permutation $\theta^* \in \mathrm{Ant}(X)$ must map edges in $\bigcup_{i=1}^{t-1} E_{v_i}$ onto edges in $\bigcup_{i=1}^{t-1} \overline{E}_{v_i}$. Thus $\hat{X} = (V \setminus \{v_1, v_2, \ldots, v_{t-1}\}, E \setminus \bigcup_{i=1}^{t-1} E_{v_i})$ is a self-complementary k-hypergraph with

$$\theta^* \in \mathrm{Ant}(\hat{X})$$

and

$$\bigcap_{i=1}^{t-1} \mathrm{Aut}(X)_{v_i} \leq \mathrm{Aut}(\hat{X}).$$

Moreover, the group $\bigcap_{i=1}^{t-1} \mathrm{Aut}(X)_{v_i}$ acts transitively on $V(\hat{X}) = V \setminus \{v_1, v_2, \ldots, v_{t-1}\}$. Hence $\hat{X}$ is a vertex transitive self-complementary k-hypergraph of order

$$|V \setminus \{v_1, v_2, \ldots, v_{t-1}\}| = n - t + 1 \equiv 1 \pmod{2^{\ell+1}},$$

and so by Theorem 4.1.1 it follows that

$$p^{(n-t+1)_{(p)}} \equiv 1 \pmod{2^{\ell+1}} \quad \text{for every prime } p.$$

∎

When $t = 2$, Theorem 4.1.3 gives necessary conditions on the order of a doubly transitive self-complementary k-hypergraph in the cases where $k = 2^\ell$ or $k = 2^\ell + 1$.

Corollary 4.1.4 *Let ℓ be a positive integer, let $k = 2^\ell$ or $k = 2^\ell + 1$, and suppose $n \equiv 2 \pmod{2^{\ell+1}}$. If there exists a doubly transitive self-complementary k-hypergraph of order n, then*

$$p^{(n-1)_{(p)}} \equiv 1 \pmod{2^{\ell+1}} \quad \textit{for every prime } p.$$

∎

The following corollary gives necessary conditions on the prime divisors of $n-1$, where n is the order of a doubly transitive self-complementary k-hypergraph satisfying the hypotheses of Corollary 4.1.4, for small values of the rank k.

Corollary 4.1.5 *Let X be a doubly transitive self-complementary k-hypergraph of order n, and let p^r be the highest power of a prime p that divides $n-1$. Then the following conditions hold.*

(a) If $k = 3$, then $p^r \equiv 1 \pmod 4$.

(b) If $k = 4$ *and* n *is even, then* $p^r \equiv 1 \pmod 8$.

(c) If $k = 5$ *and* $n \equiv 2 \pmod 8$*, then* $p^r \equiv 1 \pmod 8$. ∎

4.2 Sufficient conditions on order

4.2.1 Previous results

In this section, we state previous results due to Rao [27], and Potočnik and Šajna [24, 25], which show that the necessary conditions of Theorem 4.1.1 are sufficient when $k \in \{2, 3\}$. We also state some other known sufficient conditions on the orders of vertex transitive and doubly transitive self-complementary k-uniform hypergraphs.

The following result is a partial converse to Theorem 4.1.1 in the case where n is a prime power. It is due to Potočnik and Šajna.

Theorem 4.2.1 [24] *There exists a vertex transitive self-complementary* k*-hypergraph of order* n *for every prime power* n *congruent to 1 modulo* $2^{\ell+1}$*, where* $\ell = \max\{k_{(2)}, (k-1)_{(2)}\}$.

Potočnik and Šajna proved Theorem 4.2.1 using a Paley k-hypergraph construction. We will generalize their construction in Section 4.2.2 and prove that the converse to Theorem 4.1.1 is true in general, and not just when n is a prime power.

Recall that a *Muzychuk integer* is a positive integer n satisfying $p^{n_{(p)}} \equiv 1 \pmod 4$ for all primes p. In [27], Rao constructed vertex transitive self-complementary graphs of order n for every Muzychuk integer n. In [24], Potočnik and Šajna used a wreath product construction to find vertex transitive self-complementary 3-hypergraphs of order n for every Muzychuk integer n. Hence the condition of Theorem 4.1.1 is sufficient when $k = 2$, or when $k = 3$ and n is odd, as the next result states.

Theorem 4.2.2 *Let n be a positive integer.*

(1) **[27]** *There exists a vertex transitive self-complementary graph of order n if and only if n is a Muzychuk integer.*

(2) **[24]** *If n is odd, there exists a vertex transitive self-complementary 3-hypergraph of order n if and only if n is a Muzychuk integer.*

In [24], Potočnik and Šajna also constructed doubly transitive self-complementary 3-hypergraphs of order $q+1$ for every prime power q congruent to 1 modulo 4. Then using a rank increasing construction and a wreath product construction, they obtained the following sufficient conditions on the order of vertex transitive and doubly transitive self-complementary k-hypergraphs.

Theorem 4.2.3 **[24]** *Let k be a positive integer, let n be a Muzychuk integer, and let q be a prime power congruent to 1 modulo 4.*

(1) If $k \equiv 2$ or $3 \pmod 4$, then there exists a self-complementary vertex transitive k-hypergraph of order n.

(2) If $k \equiv 3 \pmod 4$, then there exist self-complementary vertex transitive k-hypergraphs of order $2n$ and order $(1+q)n$, and there exists a doubly transitive self-complementary k-hypergraph of order $1+q$.

4.2.2 New results - Paley uniform hypergraphs

In this section, we present a construction for a vertex transitive self-complementary uniform hypergraph of order n for every integer n satisfying the necessary conditions of Theorem 4.1.1, and consequently prove that these necessary conditions are also sufficient.

We begin with a construction for vertex transitive self-complementary uniform hypergraphs of prime power order, which is an extension of a construction due to

Potočnik and Šajna [24] for objects which they named Paley k-uniform hypergraphs. Their construction is an extension of the well-known construction of Paley graphs which can be found in Rao [27]. It should be noted that the extension to Paley 3-hypergraphs had been previously introduced by Kocay [19]. Peisert [22] also presented this construction in the case where $k = 2$ and r is any divisor of $(q-1)/4$.

If $\mathbb{F}$ is a finite field and $a_1, a_2, \ldots, a_k \in \mathbb{F}$, the *Van der Monde determinant* of $a_1, a_2, \ldots, a_k$ is defined as $VM(a_1, \ldots, a_k) = \prod_{i>j}(a_i - a_j)$.

Construction 4.2.4 Paley k-uniform hypergraph
Let k be an integer, $k \geq 2$, and let q be a prime power such that $q \equiv 1 \pmod{2^{\ell+1}}$, where $\ell = \max\{k_{(2)}, (k-1)_{(2)}\}$. Let r be a divisor of the integer $(q-1)/2^{\ell+1}$. Let $\mathbb{F}_q$ be the field of order q, let ω be a generator of the multiplicative group $\mathbb{F}_q^*$, and let $c = \gcd(q-1, r\binom{k}{2})$. For $i = 0, 1, \ldots, 2c-1$, let F_i denote the coset $\omega^i \left\langle \omega^{2r\binom{k}{2}} \right\rangle$ in $\mathbb{F}_q^*$. Finally, define $P_{q,k,r}$ to be the k-hypergraph with vertex set

$$V(P_{q,k,r}) = \mathbb{F}_q$$

and edge set

$$E(P_{q,k,r}) = \{\{a_1, \ldots, a_k\} \in \mathbb{F}_q^{(k)} : VM(a_1, \ldots, a_k) \in F_0 \cup \cdots \cup F_{c-1}\}.$$

Definition 4.2.5 *For a prime power q, an element $a \in \mathbb{F}_q^*$, and an element $b \in \mathbb{F}_q$, we define the mapping $\alpha_{a,b} : \mathbb{F}_q \to \mathbb{F}_q$ by $x^{\alpha_{a,b}} = ax + b$ for all $x \in \mathbb{F}_q$.*

Lemma 4.2.6 *Let $P_{q,k,r}$ be the Paley k-hypergraph of Construction 4.2.4, and let $c = \gcd(q-1, r\binom{k}{2})$.*

(1) The edge set of $P_{q,k,r}$ is well defined.

(2) Let s be an integer such that $s\binom{k}{2}$ is an odd multiple of c. Then

(a) $\langle \alpha_{\omega^{2s},0}, \alpha_{1,1} \rangle \leq Aut(P_{q,k,r})$.

(b) $\langle \alpha_{\omega^s,0}, \alpha_{1,1} \rangle \leq Aut(P_{q,k,r}) \cup Ant(P_{q,k,r})$.

(3) $\langle \alpha_{\omega^{2r},0}, \alpha_{1,1} \rangle \leq Aut(P_{q,k,r})$ *and* $\langle \alpha_{\omega^r,0}, \alpha_{1,1} \rangle \leq Aut(P_{q,k,r}) \cup Ant(P_{q,k,r})$.

(4) $\{\alpha_{a,b} : a \in \mathbb{F}_q^*, b \in \mathbb{F}_q\} \cap (Aut(P_{q,k,r}) \cup Ant(P_{q,k,r})) = \langle \alpha_{\omega^{s'},0}, \alpha_{1,1} \rangle$, *where* $s' = \gcd\{s : s \in \{1, 2, \ldots, q-1\}, s\binom{k}{2}$ *is a multiple of* $c\}$.

Proof:

(1) Since r divides $(q-1)/2^{\ell+1}$, we have $q - 1 = 2^{\ell+1}rt$ for some positive integer t. Let d be the order of $\omega^{r\binom{k}{2}}$ in $\mathbb{F}_q^*$. Then

$$d = \frac{q-1}{\gcd(q-1, r\binom{k}{2})} = \frac{q-1}{c}.$$

First consider the case when k is even. Then $k = 2^\ell k'$ for k' odd. Hence

$$\begin{aligned} d &= \frac{2^{\ell+1}rt}{\gcd(2^{\ell+1}rt, rk(k-1)/2)} = \frac{2^{\ell+1}t}{\gcd(2^{\ell+1}t, 2^\ell k'(k-1)/2)} \\ &= \frac{2^{\ell+1}t}{\gcd(2^{\ell+1}t, 2^{\ell-1}k'(k-1))} = 4\left(\frac{t}{\gcd(4t, k'(k-1))}\right). \end{aligned}$$

Since k' and $k-1$ are both odd integers, it follows that $\gcd(4t, k'(k-1))$ is a divisor of t, and so $t/\gcd(4t, k'(k-1))$ is an integer. Thus d is divisible by 4 when k is even. Now suppose that k is odd. Then $k - 1 = 2^\ell k'$ where k' is odd. We similarly obtain

$$d = 4\left(\frac{t}{\gcd(4t, kk')}\right), \tag{4.2.1}$$

and since k and k' are both odd, it follows that d is divisible by 4 when k is odd also.

Thus d is divisible by 4, and consequently the subgroup $\left\langle \omega^{2r\binom{k}{2}} \right\rangle$ is of even order and even index in $\mathbb{F}_q^*$. Hence $-1 \in \left\langle \omega^{2r\binom{k}{2}} \right\rangle$. Thus the Van der Monde determinant of an edge is well defined.

The number of distinct cosets of $\langle \omega^{2r\binom{k}{2}} \rangle$ in $\mathbb{F}_q^*$ is $(q-1)/|\omega^{2r\binom{k}{2}}| = \gcd(q-1, 2r\binom{k}{2}) = 2c$, since $\gcd(q-1, r\binom{k}{2}) = c$ and $d = (q-1)/c$ is divisible by 4, and hence even. Thus $F_0, F_1, \ldots, F_{2c-1}$ are all of the cosets of $\langle \omega^{2r\binom{k}{2}} \rangle$ in $\mathbb{F}_q^*$, and so the sets

$$A = \bigcup_{i=0}^{c-1} F_i \quad \text{and} \quad \bar{A} = \bigcup_{i=c}^{2c-1} F_i$$

partition $\mathbb{F}_q^*$. Hence the edge set of $P_{q,k,r}$ is well defined.

(2) Since $\left|\omega^{2r\binom{k}{2}}\right| = (q-1)/2c$, and the cyclic subgroup of $\mathbb{F}_q^*$ of order $(q-1)/2c$ is unique, it follows that $\left\langle \omega^{2r\binom{k}{2}} \right\rangle = \langle \omega^{2c} \rangle$. Hence $F_i = \omega^i \langle \omega^{2c} \rangle$ for all $i = 0, 1, \ldots, 2c-1$, and consequently

$$\omega^i F_j = \omega^i \omega^j \langle \omega^{2c} \rangle = \omega^{(i+j)_{[2c]}} \langle \omega^{2c} \rangle = F_{(i+j)_{[2c]}}.$$

Now if z is an integer, then

$$\omega^{(2z+1)c} A = \bigcup_{i=0}^{c-1} \omega^{(2z+1)c} F_i = \bigcup_{i=0}^{c-1} F_{(i+(2z+1)c)_{[2c]}} = \bigcup_{i=0}^{c-1} F_{i+c} = \bigcup_{i=c}^{2c-1} F_i = \bar{A},$$

and so

$$\omega^{(2z+1)c} A = \bar{A} \text{ and } \omega^{(2z+1)c} \bar{A} = A \quad \text{for every integer } z. \tag{4.2.2}$$

On the other hand,

$$\omega^{(2z)c} A = \bigcup_{i=0}^{c-1} \omega^{(2z)c} F_i = \bigcup_{i=0}^{c-1} F_{(i+2zc)_{[2c]}} = \bigcup_{i=0}^{c-1} F_i = A,$$

and so

$$\omega^{2zc} A = A \text{ and } \omega^{2zc} \bar{A} = \bar{A} \quad \text{for every integer } z. \tag{4.2.3}$$

Finally, if t is not a multiple of c, say $t = cz + j$ where $0 < j < c$, then

$$\omega^t A = \bigcup_{i=0}^{c-1} \omega^t F_i = \bigcup_{i=0}^{c-1} F_{(i+t)_{[2c]}} = \bigcup_{i=0}^{c-1} F_{(i+j+zc)_{[2c]}}$$

contains some cosets F_i with $0 \leq i \leq c-1$ and some cosets F_i with $c \leq i \leq 2c-1$. Hence

$$\omega^t A \cap A \neq \emptyset \text{ and } \omega^t A \cap \bar{A} \neq \emptyset \quad \text{if } t \text{ is not a multiple of } c. \tag{4.2.4}$$

Now let s be an integer such that $s\binom{k}{2}$ is an odd multiple of c. Then (4.2.2) implies that

$$\omega^{(2z+1)s\binom{k}{2}} A = \bar{A} \quad \text{and} \quad \omega^{(2z+1)s\binom{k}{2}} \bar{A} = A, \tag{4.2.5}$$

and (4.2.3) implies that

$$\omega^{(2z)s\binom{k}{2}} A = A \quad \text{and} \quad \omega^{(2z)s\binom{k}{2}} \bar{A} = \bar{A}, \tag{4.2.6}$$

for every integer z.

Observe that for a k-subset $\{a_1, a_2, \ldots, a_k\} \in \mathbb{F}_q^{(k)}$, an integer z, and an element $b \in \mathbb{F}_q$, we have

$$VM(\omega^t a_1 + b, \ldots, \omega^t a_k + b) = \omega^{t\binom{k}{2}} VM(a_1, \ldots, a_k). \tag{4.2.7}$$

(a) Equations (4.2.7) and (4.2.6) imply that the permutation $\alpha_{\omega^{2zs},b}$ maps the Van der Monde determinant of an element of $\mathbb{F}_q^{(k)}$ from A to A, or from $\bar{A}$ to $\bar{A}$. It follows that $\alpha_{\omega^{2zs},b}$ is an automorphism of $P_{q,k,r}$. We conclude that $\langle \alpha_{\omega^{2s},0}, \alpha_{1,1} \rangle \leq \mathrm{Aut}(P_{q,k,r})$.

(b) Equations (4.2.7) and (4.2.5) imply that the permutation $\alpha_{\omega^{(2z+1)s},b}$ maps the Van der Monde determinant of an element of $\mathbb{F}_q^{(k)}$ from A to $\bar{A}$, or vice versa. It follows that $\alpha_{\omega^{(2z+1)s},b}$ is an antimorphism of $P_{q,k,r}$. We conclude that $\{\alpha_{\omega^{(2z+1)s},b} : z \in \mathbb{Z},\ b \in \mathbb{F}_q\} \subseteq \mathrm{Ant}(P_{q,k,r})$. This implies that $\langle \alpha_{\omega^s,0}, \alpha_{1,1} \rangle \leq \mathrm{Aut}(P_{q,k,r}) \cup \mathrm{Ant}(P_{q,k,r})$.

(3) Since $\langle \omega^{r\binom{k}{2}} \rangle = \langle \omega^c \rangle$, we have $r\binom{k}{2} = mc$ for an integer m such that $\gcd((q-1)/c, m) = 1$. It was shown in Part (1) that $(q-1)/c$ is divisible by 4, and hence even, and so it follows that the integer m must be odd. Hence $r\binom{k}{2}$ is an odd multiple of c, and so the result follows from Part (2).

(4) Let $S = \{s \in \{1, 2, \ldots, q-1\} : s\binom{k}{2}$ is a multiple of $c\}$. Part (2) implies that

$$\langle \alpha_{\omega^s,0}, \alpha_{1,1} \rangle \leq \mathrm{Aut}(P_{q,k,r}) \cup \mathrm{Ant}(P_{q,k,r})$$

for all $s \in S$. It follows that

$$\{\alpha_{a,b} : a \in \langle \omega^s : s \in S \rangle, b \in \mathbb{F}_q\} \leq \mathrm{Aut}(P_{q,k,r}) \cup \mathrm{Ant}(P_{q,k,r}). \tag{4.2.8}$$

But $\langle \omega^s : s \in S \rangle$ is a cyclic group generated by $\omega^{s'}$, where $s' = \gcd\{s : s \in S\}$. Hence (4.2.8) implies that

$$\langle \alpha_{\omega^{s'},0}, \alpha_{1,1} \rangle \leq \mathrm{Aut}(P_{q,k,r}) \cup \mathrm{Ant}(P_{q,k,r}).$$

It remains to show that if $\alpha_{a,b} \in \mathrm{Aut}(P_{q,k,r}) \cup \mathrm{Ant}(P_{q,k,r})$, then $a \in \langle \omega^{s'} \rangle$. Suppose, for the sake of contradiction, that $\alpha_{a,b} \in \mathrm{Aut}(P_{q,k,r}) \cup \mathrm{Ant}(P_{q,k,r})$ but $a \notin \langle \omega^{s'} \rangle$. Now $\langle \omega^s \rangle \leq \langle \omega^{s'} \rangle$ for all $s \in S$. If $a = \omega^m$ for an integer m such that $m\binom{k}{2}$ is a multiple of c, then $m \in S$, and so $a \in \langle \omega^{s'} \rangle$, giving a contradiction. Hence we may assume that $a = \omega^n$ for an integer n such that $n\binom{k}{2}$ is *not* a multiple of c. Then (4.2.4) implies that $\omega^{n\binom{k}{2}} A \neq A$ and $\omega^{n\binom{k}{2}} A \neq \bar{A}$, and so (4.2.7) implies that $\alpha_{a,b} \notin \mathrm{Aut}(P_{q,k,r}) \cup \mathrm{Ant}(P_{q,k,r})$, giving a contradiction. We conclude that $\{\alpha_{a,b} : a \in \mathbb{F}_q^*, b \in \mathbb{F}_q\} \cap (\mathrm{Aut}(P_{q,k,r}) \cup \mathrm{Ant}(P_{q,k,r})) = \langle \alpha_{\omega^{s'},0}, \alpha_{1,1} \rangle$.

∎

In Chapter 5, we will use Lemma 4.2.6 along with results from finite permutation group theory to determine the complete automorphism group of the Paley k-hypergraph $P_{p,k,r}$ of Construction 4.2.4 for the cases in which p is prime and $k = 2^\ell$ or $k = 2^\ell + 1$.

Lemma 4.2.7 *The Paley k-hypergraph $P_{q,k,r}$ defined in Construction 4.2.4 is vertex transitive and self-complementary.*

Proof: Lemma 4.2.6(3) shows that $\langle \alpha_{1,1} \rangle \leq \mathrm{Aut}(P_{q,k,r})$. Since $\langle \alpha_{1,1} \rangle$ acts transitively on $\mathbb{F}_q$, so does $\mathrm{Aut}(P_{q,k,r})$. Hence $P_{q,k,r}$ is vertex transitive. Lemma 4.2.6(3) also shows that $Ant(P_{q,k,r}) \neq \emptyset$, and thus $P_{q,k,r}$ is self-complementary. ∎

It should be noted that Lemma 4.2.7 was first proved in 1985 for the cases with $k = 2$, $r = 1$ by Rao [27]. In 1992, Kocay [19] proved it for the cases with $k = 3$, $r = 1$, and in 2007, Potočnik and Šajna [24] proved it for all k when $r = 1$.

Construction 4.2.4 and Lemma 4.2.7 together prove the partial converse to Theorem 4.1.1 which is stated in Theorem 4.2.1, and is due to Potočnik and Šajna [24].

We can generalize Construction 4.2.4 to construct vertex transitive self-complementary k-hypergraphs of order n for all integers $n \equiv 1 \pmod{2^{\ell+1}}$ when $\ell = \max\{m_{(2)} : 1 \leq m \leq k\}$, which implies that the converse of Theorem 4.1.1 is true in general.

Construction 4.2.8 Generalized Paley k-uniform hypergraph
Let k be an integer, $k \geq 2$, and let n be a positive integer such that

$$p^{n_{(p)}} \equiv 1 \pmod{2^{\ell+1}} \quad \text{for every prime } p,$$

where ℓ is the largest positive integer such that 2^ℓ divides a positive integer m with $m \leq k$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ be the unique prime factorization of n, where p_i is prime, $\alpha_i \geq 1$ and $p_1 < p_2 < \ldots < p_t$. For each $i \in \{1, 2, \ldots, t\}$, let $q_i = p_i^{\alpha_i}$, let r_i be a divisor of the integer $(q_i - 1)/2^{\ell+1}$, and let $r = (r_1, r_2, \ldots, r_t)$. Let $\mathbb{F}_{q_i}$ denote the field of order q_i.

Let

$$V = \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \times \ldots \times \mathbb{F}_{q_{t-1}} \times \mathbb{F}_{q_t}.$$

Define a mapping $\zeta : V^{(k)} \to \mathbb{Z}_2$ by

$$\zeta(\{x_1, x_2, \ldots, x_k\}) = \begin{cases} 0, & \text{if } \{x_{1j}, x_{2j}, \ldots, x_{kj}\} \in E(P_{q_j, m, r_j})\}, \\ & \quad \text{where } j = \min\{i : 1 \leq i \leq t, |\{x_{1i}, x_{2i}, \ldots, x_{ki}\}| > 1\} \\ & \quad \text{and } m = |\{x_{1j}, x_{2j}, \ldots, x_{kj}\}|. \\ 1, & \text{otherwise.} \end{cases}$$

Now define $X_{n,k,r}$ to be the k-hypergraph with vertex set V and edge set

$$E = \{\{x_1, x_2, \ldots, x_k\} \in V^{(k)} : \zeta(\{x_1, x_2, \ldots, x_k\}) = 0\}.$$

Note that when $t = 1$ and $n = q_1 = p_1^{\alpha_1}$ is a prime power congruent to 1 modulo $2^{\ell+1}$, the k-hypergraph $X_{n,k,r}$ of Construction 4.2.8 is the same as the k-hypergraph P_{q_1,k,r_1} given by Construction 4.2.4.

Lemma 4.2.9 *The k-hypergraph $X_{n,k,r}$ defined in Construction 4.2.8 is vertex transitive and self-complementary.*

Proof: Since $p^{n_{(p)}} \equiv 1 \pmod{2^{\ell+1}}$ for every prime p, it follows that for each i, we have $q_i \equiv 1 \pmod{2^{\ell+1}}$, and hence $q_i \equiv 1 \pmod{2^{j+1}}$ for all $j \leq \ell$. Now by definition, $\ell = \max\{\ell_m : 1 < m \leq k\}$ where $\ell_m = \max\{m_{(2)}, (m-1)_{(2)}\}$. Hence $q_i \equiv 1 \pmod{2^{\ell_m+1}}$ for $m = 2, 3, \ldots, k$, and so P_{q_i,m,r_i} is well-defined for $i = 1, 2, \ldots, t$ and $m = 2, 3, \ldots, k$. Thus the edges of $X_{n,k,r}$ are well-defined.

Let $\mathbb{F}_{q_i}^*$ denote the (cyclic) multiplicative group of non-zero elements in $\mathbb{F}_{q_i}$, and let ω_i be a generator of $\mathbb{F}_{q_i}^*$. For each $i \in \{1, 2 \ldots, t\}$, an element $a \in \mathbb{F}_{q_i}^*$, and an element $b \in \mathbb{F}_{q_i}$, let $\alpha_{i,a,b}$ denote the permutation $\alpha_{a,b}$ of $\mathbb{F}_{q_i}$ defined on page 63. Then by Lemma 4.2.6(3), $\alpha_{i,\omega_i^{r_i},0} \in \mathrm{Ant}(P_{q_i,m,r_i})$ for $m = 2, 3, \ldots, k$, so it follows from the definition of $X_{n,k,r}$ that

$$\alpha_{1,\omega_1^{r_1},0} \times \cdots \times \alpha_{t,\omega_t^{r_t},0} \in \mathrm{Ant}(X_{n,k,r}).$$

Hence $X_{n,k,r}$ is self-complementary.

To see that $X_{n,k,r}$ is vertex transitive, it suffices to show that an automorphism can map the vertex $\mathbf{0} = (0, 0, \ldots, 0)$ to any other vertex. For $i = 1, 2, \ldots, t$ and for any $x_i \in \mathbb{F}_{q_i}$, the bijection $\alpha_{i,1,x_i}$ maps 0 to x_i. Moreover, Lemma 4.2.6(2) implies that $\alpha_{i,1,x_i}$ is an automorphism of P_{q_i,m,r_i}, for $m = 2, 3, \ldots, k$. Now let

$\mathbf{x} = (x_1, x_2, \ldots, x_t) \in V$. It follows from the definition of $X_{n,k,r}$ that $\alpha_{\mathbf{x}} = \alpha_{1,1,x_1} \times \alpha_{2,1,x_2} \times \cdots \times \alpha_{t,1,x_t} \in \mathrm{Aut}(X_{n,k,r})$. Since $\alpha_{\mathbf{x}}$ maps $\mathbf{0}$ to $\mathbf{x}$ and $\mathbf{x}$ was an arbitrary element of V, it follows that $\mathrm{Aut}(X_{n,k,r})$ acts transitively on V, and so $X_{n,k,r}$ is vertex transitive. ∎

Theorem 4.2.10 *Let ℓ be a positive integer, let $k = 2^\ell$ or $k = 2^\ell + 1$, and let $n \equiv 1 \pmod{2^{\ell+1}}$. There exists a vertex transitive self-complementary k-hypergraph of order n if and only if*

$$p^{n_{(p)}} \equiv 1 \pmod{2^{\ell+1}} \quad \textit{for every prime } p. \tag{4.2.9}$$

Proof: The necessity of condition (4.2.9) follows directly from Theorem 4.1.1. Since $k = 2^\ell$ or $k = 2^\ell + 1$, for any integer m such that $1 < m \leq k$, ℓ is greater than or equal to the largest integer i such that 2^i divides m. Thus k, ℓ, and n satisfy the hypotheses of Construction 4.2.8, and so the sufficiency of condition (4.2.9) follows from Lemma 4.2.9. ∎

4.3 Open problems

In [24], Potočnik and Šajna proposed the following problem for vertex transitive self-complementary uniform hypergraphs:

Problem 4.3.1 *For a given integer $k \geq 2$, determine the set $\mathcal{M}_k$ of all integers n for which there exists a vertex transitive self-complementary k-hypergraph of order n.*

(Note that the set $\mathcal{M}_k$ is a subset of the set $\mathcal{L}_{1,k}$ of Problem 3.2.8 for all $k \geq 2$. Hence any partial solution to Problem 4.3.1 provides a partial solution to Problem 3.2.8 for $t = 1$, but not conversely.)

Theorem 4.2.2 gives the solution to Problem 4.3.1 when $k = 2$, and it gives all odd integers in $\mathcal{M}_3$. Theorem 4.2.3 and Theorem 4.2.10 give some subsets of $\mathcal{M}_k$ for $k = 2^\ell$ and $k = 2^\ell + 1$. However, Problem 4.3.1 remains mainly unanswered, and the problem seems to have no simple solution. In [24], Potočnik and Šajna suggest the following more feasible subproblem.

The first k for which Problem 4.3.1 remains unsolved is $k = 3$. Theorem 4.2.2(2) gives a solution for all odd orders n. For even orders, Theorem 3.1.4 implies that if $2m \in \mathcal{M}_3$, then m is odd. Moreover, Theorem 4.2.3(2) implies that $2m \in \mathcal{M}_3$ for every Muzychuk integer m, but it also implies that $(q + 1)m \in \mathcal{M}_3$ for every Muzychuk integer m and every prime power q congruent to 1 modulo 4. Hence there do exist non-Muzychuk integers m such that $2m \in \mathcal{M}_k$. What form can such integers m take? Potočnik and Šajna posed the following problem.

Problem 4.3.2 *Find all odd non-Muzychuk integers m such that there exists a vertex transitive self-complementary 3-hypergraph of order $2m$.*

One natural first step for solving Problem 4.3.2 is to determine the primes p for which $2p \in \mathcal{M}_3$. To do this, it would be useful to know something about the structure of a vertex transitive self-complementary k-hypergraph of prime order. To that end, in Chapter 5 we use group-theoretic results due to Burnside and Zassenhaus to determine the automorphisms and antimorphisms of these objects. Then we use this information to generate all such hypergraphs under certain conditions.

Chapter 5

Transitive self-complementary hypergraphs of prime order

In this chapter, we determine the automorphisms and antimorphisms of the vertex transitive self-complementary k-hypergraphs of prime order p in the case where $p \equiv 1 \pmod{2^{\ell+1}}$ and $k = 2^\ell$ or $k = 2^\ell + 1$, and we present an algorithm for generating all of these structures. As a consequence, we obtain a bound on the number of pairwise non-isomorphic vertex transitive self-complementary graphs of prime order $p \equiv 1 \pmod 4$.

5.1 Preliminaries - some group theory

In this section, we introduce some notation, and a couple of preliminary group-theoretic results.

For a prime p, let $\mathbb{F}_p^*$ denote the multiplicative group of units of the finite field $\mathbb{F}_p$ of order p. Given $a \in \mathbb{F}_p^*$ and $b \in \mathbb{F}_p$, define the mapping $T_{a,b} : \mathbb{F}_p \to \mathbb{F}_p$ by $T_{a,b} : x \mapsto ax + b$. One can show that $T_{a,b}$ is a permutation of $\mathbb{F}_p$, and that $\{T_{a,b} : a \in \mathbb{F}_p^*\ ,\ b \in \mathbb{F}_p\}$ is a group, called the *affine linear group of permutations* acting on $\mathbb{F}_p$.

This group will be denoted by $AGL_1(p)$.

If H is a subgroup of a group G, we will denote this by $H \leq G$. Two permutation groups $G \leq Sym(\Omega)$ and $H \leq Sym(\Pi)$ are *equivalent* if there exist bijections $\alpha : \Omega \rightarrow \Pi$ and $\beta : G \rightarrow H$ such that

$$g : v \mapsto w \iff g^{\beta} : v^{\alpha} \mapsto w^{\alpha},$$

for all $g \in G$ and all $v, w \in \Omega$, and we denote this by $G \equiv H$. A permutation group G acting on a finite set Ω is *sharply transitive* if for any two points $v, w \in \Omega$, there is exactly one permutation $g \in G$ such that $v^g = w$. The group G is *sharply doubly transitive* if G is sharply transitive in its action on ordered pairs of distinct elements from Ω.

The following two theorems due to Burnside [35] and Zassenhaus [38] will be used to restrict the automorphism group of a vertex transitive k-hypergraph of prime order.

Theorem 5.1.1 **[35]** *If G is a transitive permutation group acting on a prime number p of elements, then either G is doubly transitive or*

$$G \equiv \{T_{a,b} : a \in H \leq \mathbb{F}_p^* \ , \ b \in \mathbb{F}_p\}.$$

Theorem 5.1.2 **[38, 11]** *A sharply doubly transitive permutation group of prime degree p is equivalent as a permutation group to $AGL_1(p)$.*

We will also require the following useful and well-known counting tool, called the *orbit-stabilizer* lemma.

Lemma 5.1.3 **[35]** *Let G be a permutation group acting on V and let x be a point in V. Then*

$$|G| = |G_x||x^G|.$$

5.2 A characterization

Now we are ready to determine the automorphisms and antimorphisms of the vertex transitive self-complementary k-hypergraphs of prime order p in the cases where $p \equiv 1 \pmod{2^{\ell+1}}$ and $k = 2^\ell$ or $k = 2^\ell + 1$.

Lemma 5.2.1 *Let ℓ be a positive integer, and suppose that $k = 2^\ell$ or $k = 2^\ell + 1$. If X is a vertex transitive self-complementary k-hypergraph of prime order $p \equiv 1 \pmod{2^{\ell+1}}$, then $Ant(X) \cup Aut(X)$ is equivalent as a permutation group to a subgroup of $AGL_1(p)$. That is*

$$Ant(X) \cup Aut(X) \equiv \{T_{a,b} : a \in G \leq \mathbb{F}_p^* \ , \ b \in \mathbb{F}_p\}.$$

Proof: Since X is vertex transitive, it follows that $\mathrm{Aut}(X)$ and $\mathrm{Ant}(X) \cup \mathrm{Aut}(X)$ are both transitive permutation groups acting on a prime number of elements. Since $p \equiv 1 \pmod{2^{\ell+1}}$, Theorem 3.1.4 implies that X is not doubly transitive, and so by Burnside's Theorem,

$$\mathrm{Aut}(X) \equiv \{T_{a,b} : a \in H \leq \mathbb{F}_p^* \ , \ b \in \mathbb{F}_p\} \tag{5.2.1}$$

for some subgroup H of $\mathbb{F}_p^*$. Now since $AGL_1(p)$ is doubly transitive and X is not doubly transitive, we have $\mathrm{Aut}(X) \not\equiv AGL_1(p)$. Hence H is a proper subgroup of $\mathbb{F}_p^*$ in Equation (5.2.1), and so $|H| \leq (p-1)/2$. Thus $|\mathrm{Aut}(X)| = p|H| \leq p(p-1)/2$. Since $\mathrm{Aut}(X)$ is an index-2 subgroup of $\mathrm{Aut}(X) \cup \mathrm{Ant}(X)$, we have $|\mathrm{Aut}(X) \cup \mathrm{Ant}(X)| = 2|\mathrm{Aut}(X)| \leq p(p-1)$.

If $\mathrm{Aut}(X) \cup \mathrm{Ant}(X)$ is not doubly transitive, then the result follows from Burnside's Theorem 5.1.1. On the other hand, if $\mathrm{Aut}(X) \cup \mathrm{Ant}(X)$ is doubly transitive, then certainly $|\mathrm{Aut}(X) \cup \mathrm{Ant}(X)| \geq p(p-1)$, which implies that $|\mathrm{Aut}(X) \cup \mathrm{Ant}(X)| = p(p-1)$. Hence $\mathrm{Aut}(X) \cup \mathrm{Ant}(X)$ must be sharply doubly transitive, and so in this case the result follows from Zassenhaus' Theorem 5.1.2. ∎

In the next lemma, we completely determine the set of automorphisms and antimorphisms of the Paley k-hypergraphs of Construction 4.2.4 which have prime order.

Lemma 5.2.2 *Let ℓ be a positive integer, and suppose that $k = 2^\ell$ or $k = 2^\ell + 1$. Let p be a prime such that $p \equiv 1 \pmod{2^{\ell+1}}$. Let r be a divisor of $(p-1)/2^{\ell+1}$, and let $X = P_{p,k,r}$ be the Paley k-hypergraph defined in Construction 4.2.4. Let S be the set of elements $s \in \{1, 2, \ldots, p-1\}$ such that $s\binom{k}{2}$ is a multiple of $c = \gcd(p-1, r\binom{k}{2})$, and let $s' = \gcd\{s : s \in S\}$. Then*

$$Aut(X) = \{T_{a,b} : a \in \langle \omega^{2s'} \rangle \leq \mathbb{F}_p^* \ , \ b \in \mathbb{F}_p\}$$

and

$$Ant(X) \cup Aut(X) = \{T_{a,b} : a \in \langle \omega^{s'} \rangle \leq \mathbb{F}_p^* \ , \ b \in \mathbb{F}_p\},$$

where ω is a generator of $\mathbb{F}_p^*$.

Proof: Since $T_{a,b}$ is equal to the permutation $\alpha_{a,b}$ defined on page 63, Lemma 4.2.6(4) guarantees that

$$\{T_{a,b} : a \in \mathbb{F}_q^*, b \in \mathbb{F}_q\} \cap (\mathrm{Aut}(X) \cup \mathrm{Ant}(X)) = \langle T_{\omega^{s'},0}, T_{1,1} \rangle.$$

On the other hand, Lemma 5.2.1 implies that

$$\mathrm{Ant}(X) \cup \mathrm{Aut}(X) \equiv \{T_{a,b} : a \in G \leq \mathbb{F}_p^* \ , \ b \in \mathbb{F}_p\}$$

for some subgroup G of $\mathbb{F}_p^*$. We conclude that

$$\mathrm{Ant}(X) \cup \mathrm{Aut}(X) = \{T_{a,b} : a \in \langle \omega^{s'} \rangle \leq \mathbb{F}_p^* \ , \ b \in \mathbb{F}_p\}.$$

Since $\mathrm{Aut}(X)$ is an index-2 subgroup of $\mathrm{Ant}(X) \cup \mathrm{Aut}(X)$, it follows that

$$\mathrm{Aut}(X) = \{T_{a,b} : a \in \langle \omega^{2s'} \rangle \leq \mathbb{F}_p^* \ , \ b \in \mathbb{F}_p\}.$$

∎

Theorem 5.2.3

Suppose $X = (V, E)$ is a vertex-transitive self-complementary k-hypergraph of prime order p, where $k = 2^\ell$ or $k = 2^\ell + 1$ and $p \equiv 1 \pmod{2^{\ell+1}}$. Let ω be a generator of $\mathbb{F}_p$, and let $r = p(p-1)/|Aut(X) \cup Ant(X)|$. Then X is isomorphic to a k-hypergraph Y with vertex set $\mathbb{F}_p$ for which $Aut(Y) = \langle T_{\omega^{2r},0}, T_{1,1}\rangle \leq Aut(P_{p,k,r})$ and $Ant(Y) \cup Aut(Y) = \langle T_{\omega^r,0}, T_{1,1}\rangle \leq Ant(P_{p,k,r}) \cup Aut(P_{p,k,r})$. Consequently, Y is in the θ-switching class of $P_{p,k,r}$ for every permutation $\theta \in \{T_{\omega^{rm},b} : m \text{ odd}, b \in \mathbb{F}_p\}$.

Proof: By Lemma 5.2.1,

$$Ant(X) \cup Aut(X) \equiv \{T_{a,b} : a \in G \leq \mathbb{F}_p^* \ , \ b \in \mathbb{F}_p\},$$

and $Aut(X)$ is an index-2 subgroup of this group, so

$$Aut(X) \equiv \{T_{a,b} : a \in K \leq \mathbb{F}_p^* \ , \ b \in \mathbb{F}_p\},$$

where K is an index-2 subgroup of G. Thus there is a bijection $\varphi : V \to \mathbb{F}_p$ such that $Y = (\varphi(V), \varphi(E))$ satisfies

$$Ant(Y) \cup Aut(Y) = \{T_{a,b} : a \in G \leq \mathbb{F}_p^* \ , \ b \in \mathbb{F}_p\},$$

and

$$Aut(Y) = \{T_{a,b} : a \in K \leq \mathbb{F}_p^* \ , \ b \in \mathbb{F}_p\}.$$

Now $|Ant(Y) \cup Aut(Y)|$ is even, and its order divides $p(p-1)$. Since

$$r = \frac{p(p-1)}{|Ant(Y) \cup Aut(Y)|} = \frac{p(p-1)}{|Ant(X) \cup Aut(X)|}$$

and ω is a generator of $\mathbb{F}_p^*$, it follows that $G = \langle \omega^r \rangle$ and $K = \langle \omega^{2r} \rangle$. If r is a divisor of $\frac{(p-1)}{2^{\ell+1}}$, then $P_{p,k,r}$ exists and $Aut(Y) = \langle T_{\omega^{2r},0}, T_{1,1}\rangle \leq Aut(P_{p,k,r})$ and $Ant(Y) \cup Aut(Y) = \langle T_{\omega^r,0}, T_{1,1}\rangle \leq Ant(P_{p,k,r}) \cup Aut(P_{p,k,r})$. Consequently, Y is in the θ-switching class of $P_{p,k,r}$ for every $\theta \in \langle T_{\omega^r,0}, T_{1,1}\rangle \setminus \langle T_{\omega^{2r},0}, T_{1,1}\rangle = \{T_{\omega^{rm},b} : m \text{ odd}, b \in \mathbb{F}_p\}$.

It remains to show that $r = \frac{p(p-1)}{|Aut(X) \cup Ant(X)|}$ is a divisor of $(p-1)/2^{\ell+1}$. First we will show that both of the integers p and 2^ℓ divide $|Aut(Y)|$. We have $Aut(Y) = \{T_{a,b} : a \in K \leq \mathbb{F}_p^*\ ,\ b \in \mathbb{F}_p\}$, which contains the subgroup $\{T_{1,b} : b \in \mathbb{F}_p\}$ of order p, and so p divides $|Aut(Y)|$. Now let $\theta \in Ant(Y)$. Then θ has even order in $Ant(Y) \cup Aut(Y)$, so $|\theta| = 2^j s$ for some positive integer j and some odd positive integer s. Now $\theta^s \in Ant(Y)$ and θ^s has order 2^j, so Lemma 2.2.8 implies that θ^s has exactly one fixed point, and all other orbits of θ^s have length divisible by $2^{\ell+1}$. Hence the order of the antimorphism θ^s is divisible by $2^{\ell+1}$, and so $|Aut(Y) \cup Ant(Y)| = 2|Aut(Y)|$ is divisible by $2^{\ell+1}$. It follows that 2^ℓ divides $|Aut(Y)|$.

Now observe that

$$\begin{aligned} r = \frac{p(p-1)}{|Aut(X) \cup Ant(X)|} &= \frac{p(p-1)}{|Aut(Y) \cup Ant(Y)|} = \frac{p(p-1)2^{\ell+1}}{2|Aut(Y)|2^{\ell+1}} \\ \implies \frac{p-1}{2^{\ell+1}} &= r\left(\frac{|Aut(Y)|}{p2^\ell}\right). \end{aligned} \tag{5.2.2}$$

Since $|Aut(Y)|$ is divisible by the odd prime p, and $|Aut(Y)|$ is also divisible by 2^ℓ, it follows that $\frac{|Aut(Y)|}{p2^\ell}$ is an integer. Hence Equation (5.2.2) implies that r divides the integer $\frac{p-1}{2^{\ell+1}}$. This completes the proof. ∎

5.3 Generating transitive k-hypergraphs

In this section, we present an algorithm for generating all vertex transitive self-complementary k-hypergraphs of prime order $p \equiv 1 \pmod{2^{\ell+1}}$ in the case where $k = 2^\ell$ or $k = 2^\ell + 1$.

Algorithm 5.3.1

Let ℓ be a positive integer, and suppose that $k = 2^\ell$ or $k = 2^\ell + 1$. Let p be a prime such that $p \equiv 1 \pmod{2^{\ell+1}}$. Let ω be a generator of $\mathbb{F}_p^*$.

1. Choose a divisor r of $(p-1)/2^{\ell+1}$, let $P_{p,k,r}$ be the Paley k-hypergraph of order p, and let $\theta = T_{\omega^r,0}$.

   (a) Take an arbitrary uncoloured element A of $\mathbb{F}_p^{(k)}$. In Steps (i), (ii) and (iii) below, we will find the orbit $\mathcal{O} = A^{\langle T_{\omega^r,0}, T_{1,1}\rangle}$ of the group $\langle T_{\omega^r,0}, T_{1,1}\rangle$ on $\mathbb{F}_p^{(k)}$ which contains A.

      (i) Create a sequence of elements of $\mathbb{F}_p^{(k)}$

$$A, \quad A^{\theta}, \quad A^{\theta^2}, \quad A^{\theta^3}, \quad \dots, \quad A^{\theta^{|\theta|-1}}. \tag{5.3.1}$$

      If $A \in E(P_{p,k,r})$, then colour the elements of the form $A^{\theta^{2i}}$ red and those of the form $A^{\theta^{2i+1}}$ blue. If $A \notin E(P_{p,k,r})$, then colour the elements of the form $A^{\theta^{2i}}$ blue and those of the form $A^{\theta^{2i+1}}$ red.

      (ii) Repeat Step 1(a)(i) but replace A with an element of $A^{\langle T_{\omega^r,0}, T_{1,1}\rangle}$ which is uncoloured.

      (iii) Repeat Step 1(a)(ii) until all elements of $A^{\langle T_{\omega^r,0}, T_{1,1}\rangle}$ have been coloured.

   (b) Repeat Step 1(a) until all of the elements of $\mathbb{F}_p^{(k)}$ have been coloured.

   (c) Let m be the number of orbits of the group $\langle T_{\omega^r,0}, T_{1,1}\rangle$ on $\mathbb{F}_p^{(k)}$ created in Steps 1(a) and 1(b), and choose an ordering $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_m$ of these orbits.

      (i) Choose a vector $v \in \mathbb{Z}_2^m$, and let X_v^r be the k-hypergraph with vertex set $\mathbb{F}_p$ and edge set E, where an edge $e \in \mathcal{O}_i$ is in E
      if and only if e is red and $v_i = 1$, or e is blue and $v_i = 0$.

      (ii) Repeat step 2(c)(i) for all vectors $v \in \mathbb{Z}_2^m$.

2. Repeat step 1 for all divisors r of $(p-1)/2^{\ell+1}$.

Lemma 5.3.2 *The colouring of the elements of $\mathbb{F}_p^{(k)}$ in Algorithm 5.3.1 is well defined.*

Proof: Lemma 4.2.6(3) guarantees that $\theta = T_{\omega^r,0}$ is an antimorphism of the Paley k-hypergraph $P_{p,k,r}$, and so θ is a k-complementing permutation. Thus Proposition 2.1.1 guarantees that no element of the sequence (5.3.1) in Step 1(a)(i) is coloured both red and blue. Also, since the orbits of θ partition the elements of $V^{(k)}$, no k-subset of V can occur in more than one sequence (5.3.1) created in Step 1(a)(i). Hence no element of $\mathbb{F}_p^{(k)}$ is coloured both red and blue in steps 1(a) and 1(b). ∎

Theorem 5.3.3 *Let ℓ be a positive integer, and suppose that $k = 2^\ell$ or $k = 2^\ell + 1$. Let p be a prime such that $p \equiv 1 \pmod{2^{\ell+1}}$. Let X be a k-hypergraph of order p. Then X is vertex transitive and self-complementary if and only if X is isomorphic to a k-hypergraph generated by Algorithm 5.3.1.*

Proof: ($\Rightarrow$) Suppose that X is a vertex transitive self-complementary k-hypergraph of order p. By Theorem 5.2.3, X is isomorphic to a k-hypergraph Y with vertex set $\mathbb{F}_p$ for which $Aut(Y) = \langle T_{\omega^{2r},0}, T_{1,1}\rangle \leq Aut(P_{p,k,r})$ and $Ant(Y) \cup Aut(Y) = \langle T_{\omega^r,0}, T_{1,1}\rangle \leq Ant(P_{p,k,r}) \cup Aut(P_{p,k,r})$, where $r = p(p-1)/|Aut(X) \cup Ant(X)|$. We will obtain Y from $P_{p,k,r}$ using Algorithm 5.3.1.

Certainly $P_{p,k,r}$ is generated by Algorithm 5.3.1, since $P_{p,k,r} = X^r_{\mathbf{1}}$. Now we will show how Y can be generated by Algorithm 5.3.1 from $P_{p,k,r}$. By Theorem 5.2.3, Y is in the θ-switching class of $P_{p,k,r}$ for every permutation $\theta \in \{T_{\omega^{rm},b} : m \text{ odd}, b \in \mathbb{F}_p\}$. In particular, Y is $T_{\omega,0}$-switching equivalent to $P_{p,k,r}$. That is, Y can be obtained from $P_{p,k,r}$ by changing edges to nonedges, and vice versa, in some collection S of orbits of $T_{\omega,0}$ on $\mathbb{F}_p^{(k)}$. Moreover, since $Aut(Y) = \langle T_{\omega^{2r},0}, T_{1,1}\rangle$, the collection S must also be equal to a union of orbits of $\langle T_{\omega^{2r},0}, T_{1,1}\rangle$ on $\mathbb{F}_p^{(k)}$. Hence S is a union of orbits of $\langle T_{\omega^r,0}, T_{\omega^{2r},0}, T_{1,1}\rangle = \langle T_{\omega^r,0}, T_{1,1}\rangle$ on $\mathbb{F}_p^{(k)}$. This implies that Y can be obtained from $P_{p,k,r}$ by changing edges to nonedges, and vice versa, in a subset $\mathcal{S}$ of the orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_m$ given by Algorithm 5.3.1. Let $v \in \mathbb{Z}_2^m$ be the vector such that $v_i = 1$

if and only if $\mathcal{O}_i \in \mathcal{S}$. Then $Y = X^r_{\mathbf{1}+v}$. Since $X \cong Y$, we have $X \cong X^r_{\mathbf{1}+v}$, and so X is isomorphic to a k-hypergraph generated by Algorithm 5.3.1.

($\Leftarrow$) Suppose that X is a k-hypergraph of order p that is isomorphic to a k-hypergraph generated by Algorithm 5.3.1. We will show that X is vertex transitive and self-complementary. Now $X \cong X^r_v$ for some divisor r of $(p-1)/2^{\ell+1}$ and some $v \in \mathbb{Z}_2^m$, where m is the number of orbits of the group $\langle T_{\omega^r,0}, T_{1,1}\rangle$ on $\mathbb{F}_p^{(k)}$. The k-hypergraph X^r_v is constructed by choosing either the red or the blue edges from each of the orbits in $\{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_m\}$. Our coloring method in Step 1(a) guarantees that each of the set of red edges and the set of blue edges in $\mathcal{O}_i$ constitutes an orbit of $\langle T_{\omega^{2r},0}, T_{1,1}\rangle$ on $\mathbb{F}_p^{(k)}$, for all $i \in \{1, 2, \ldots, m\}$. This implies that $\langle T_{\omega^{2r},0}, T_{1,1}\rangle \leq Aut(X^r_v)$. Since $\langle T_{1,1}\rangle \leq \langle T_{\omega^{2r},0}, T_{1,1}\rangle$, and $\langle T_{1,1}\rangle$ acts transitively on $\mathbb{F}_p$, we conclude that $Aut(X^r_v)$ acts transitively on $V(X^r_v) = \mathbb{F}_p$, and so X^r_v is vertex transitive. Our coloring method in Step 1(a) also guarantees that $T_{\omega^r,0}$ maps red edges onto blue edges, and vice versa, in the orbit $\mathcal{O}_i$, for all $i \in \{1, 2, \ldots, m\}$. This implies that $T_{\omega^r,0} \in Ant(X^r_v)$, and so X^r_v is self-complementary.

Hence X^r_v is a vertex transitive self-complementary k-hypergraph of order p, and since $X \cong X^r_v$, so is X. ∎

When $k = 2$ or $k = 3$, Theorem 3.1.4 guarantees that for every vertex transitive self-complementary k-hypergraph of prime order p, we must have $p \equiv 1 \pmod 4$. Hence Algorithm 5.3.1 generates *every* vertex transitive self-complementary graph and 3-hypergraph of prime order. In addition, for the case $k = 2$ it is possible to count the orbits of the group $\langle T_{\omega^r,0}, T_{1,1}\rangle$ on $\mathbb{F}_p^{(k)}$, and consequently we obtain a bound on the number of pairwise non-isomorphic vertex transitive self-complementary graphs of order p.

Corollary 5.3.4 *For any prime $p \equiv 1 \pmod 4$, there are at most*

$$\sum_{r \mid \frac{p-1}{4}} 2^{r-1}$$

pairwise non-isomorphic vertex transitive self-complementary graphs of order p.

Proof: Let r be a divisor of $(p-1)/4$. Then $\gcd(p-1, r) = r$. Let ω be a generator of $\mathbb{F}_p$. For each $i = 0, 1, \ldots, 2r-1$, let $\mathcal{E}_i = \{e \in \mathbb{F}_p^{(2)} : VM(e) \in \omega^i \langle \omega^{2r} \rangle\}$. We will prove that each of the orbits of the group $\langle T_{\omega^r,0}, T_{1,1} \rangle$ on $\mathbb{F}_p^{(2)}$ has the form $\mathcal{E}_i \cup \mathcal{E}_{i+r}$, for some $i = 0, 1, \ldots, r-1$. For a given divisor r of $(p-1)/4$, Algorithm 5.3.1 generates at most 2^{m-1} pairwise non-isomorphic graphs X with $\mathrm{Aut}(X) \cup \mathrm{Ant}(X) = \langle T_{\omega^r,0}, T_{1,1} \rangle$, where m is the number of orbits of $\langle T_{\omega^r,0}, T_{1,1} \rangle$ on $\mathbb{F}_p^{(2)}$. Finding these orbits explicitly will lead us to conclude that $m = r$ for each divisor r of $(p-1)/4$, and so the result will follow.

First we show that each element of $\{T_{\omega^{rm},b} : m \text{ odd}, b \in \mathbb{F}_p\}$ maps edges of $\mathcal{E}_i$ to edges of $\mathcal{E}_{i+r}$, where addition of subscripts is addition modulo $2r$. Let $\theta = T_{\omega^{rm},b}$ for some odd integer m and some $b \in \mathbb{F}_p$. Now if $\{x, y\} \in \mathcal{E}_i$, then $VM(\{x, y\}) \in \omega^i \langle \omega^{2r} \rangle$. Hence $VM(\{x, y\}^\theta) = x^\theta - y^\theta = \omega^{mr}(x - y) = \omega^{mr} VM(\{x, y\}) \in \omega^{i+r} \langle \omega^{2r} \rangle$. Thus $\{x, y\}^\theta \in \mathcal{E}_{i+r}$.

Now let $G = \langle T_{\omega^{2r},0}, T_{1,1} \rangle$. We show that each element of G maps edges of $\mathcal{E}_i$ to edges of $\mathcal{E}_i$. Let $\alpha \in G$. Let $\alpha = T_{\omega^{2rm},b}$ for some integer m and some $b \in \mathbb{F}_p$. Now if $\{x, y\} \in \mathcal{E}_i$, then $VM(\{x, y\}) \in \omega^i \langle \omega^{2r} \rangle$. Hence $VM(\{x, y\}^\alpha) = x^\alpha - y^\alpha = \omega^{2rm}(x - y) = \omega^{2rm} VM(\{x, y\}) \in \omega^i \langle \omega^{2r} \rangle$. Thus $\{x, y\}^\alpha \in \mathcal{E}_i$. Hence $\mathcal{E}_i^\alpha = \mathcal{E}_i$, for all $\alpha \in G$.

Hence $G = \langle T_{\omega^{2r},0}, T_{1,1} \rangle$ maps edges of $\mathcal{E}_i$ onto edges of $\mathcal{E}_i$, and $\{T_{\omega^r m,b} : m \text{ odd}, b \in \mathbb{F}_p\}$ maps edges of $\mathcal{E}_i$ to edges of $\mathcal{E}_{i+r}$. This implies that each orbit of $\langle T_{\omega^r,0}, T_{1,1} \rangle$ on $\mathbb{F}_p^{(2)}$ is contained in $\mathcal{E}_i \cup \mathcal{E}_{i+r}$, for some $i \in \{0, 1, \ldots, r-1\}$.

Next we prove that $\langle T_{\omega^r,0}, T_{1,1} \rangle$ acts transitively on $\mathcal{E}_i \cup \mathcal{E}_{i+r}$, for all $i = 0, 1, \ldots, r-1$. It suffices to show that $G = \langle T_{\omega^{2r},0}, T_{1,1} \rangle$ acts transitively on the set of edges $\mathcal{E}_i$, for all $i = 0, 1, \ldots, 2r-1$. Since $G = \langle T_{\omega^{2r},0}, T_{1,1}\rangle$, we have $|G| = p(p-1)/2r$. Now fix $\{x, y\} \in \mathbb{F}_p^{(2)}$. Recall that the group $AGL_1(p)$ acting on $\mathbb{F}_p$ is sharply doubly transitive. Since $G \leq AGL_1(p)$, it follows that at most two permutations in G fix $\{x, y\}$. Hence by the orbit-stabilizer Lemma 5.1.3, we obtain

$$|\{x, y\}^G| = |G|/|G_{\{x,y\}}| \geq |G|/2 = p(p-1)/4r. \tag{5.3.2}$$

Also, for integers i and j such that $0 \leq i, j \leq 2r-1$, we have $|\mathcal{E}_i| = |\mathcal{E}_j|$. This implies that each of the edge sets $\mathcal{E}_i$ has size

$$|\mathcal{E}_i| = |\mathbb{F}_p^{(2)}|/2r = p(p-1)/4r. \tag{5.3.3}$$

Now (5.3.2) and (5.3.3) together imply that

$$|\{x, y\}^G| \geq |\mathcal{E}_i|, \quad \text{for all } i \in \{0, 1, \ldots, 2r-1\}. \tag{5.3.4}$$

Since each orbit of G on $\mathbb{F}_p^{(2)}$ is contained in $\mathcal{E}_i$ for some i, and (5.3.4) implies that each orbit of G on $\mathbb{F}_p^{(2)}$ has cardinality at least $|\mathcal{E}_i|$ for all i, it follows that each orbit of G on $\mathbb{F}_p^{(2)}$ is equal $\mathcal{E}_i$ for some i. Hence G acts transitively on the set of edges $\mathcal{E}_i$, for all $i = 0, 1, \ldots, 2r-1$. This implies that $\langle T_{\omega^r,0}, T_{1,1}\rangle$ acts transitively on $\mathcal{E}_i \cup \mathcal{E}_{i+r}$, for all $i = 0, 1, \ldots, r-1$.

Since each orbit of $\langle T_{\omega^r,0}, T_{1,1}\rangle$ is contained in $\mathcal{E}_i \cup \mathcal{E}_{i+r}$ for some i, the fact that $\langle T_{\omega^r,0}, T_{1,1}\rangle$ acts transitively on $\mathcal{E}_i \cup \mathcal{E}_{i+r}$ implies that each orbit of $\langle T_{\omega^r,0}, T_{1,1}\rangle$ on $\mathbb{F}_p^{(2)}$ is equal to $\mathcal{E}_i \cup \mathcal{E}_{i+r}$ for some $i = 0, 1, \ldots, r-1$. There are exactly r such orbits, and so $m = r$ in step 1(c) of Algorithm 5.3.1. Thus for each divisor r of $(p-1)/4$, Algorithm 5.3.1 generates exactly $|\mathbb{Z}_2^r| = 2^r$ vertex transitive self-complementary graphs of order p. Now every graph generated by the algorithm is isomorphic to its complement, which is also generated by the algorithm. It follows that there are at most

$$\sum_{r \mid \frac{p-1}{4}} 2^{r-1}$$

pairwise non- isomorphic vertex transitive self-complementary graphs of order p. ∎

5.4 Open problems

In this section, the author proposes two open problems.

When neither k nor $k-1$ is a power of 2, not much is known about the structure of vertex transitive k-uniform hypergraphs of prime order p. However, using Burnside's Theorem, one may solve the following problem by examining the structure of doubly transitive permutation groups.

Problem 5.4.1 *Let p be prime, and let k be a positive integer, $k \leq p-1$. Characterize the structure of vertex transitive self-complementary k-uniform hypergraphs of order p.*

In [9], Dobson proved the following analogue to Burnside's characterization of transitive groups of prime degree for transitive groups of prime power degree.

Theorem 5.4.2 [9] *A transitive group of odd prime-power degree such that every minimal transitive subgroup is cyclic is either doubly transitive (and hence known) or contains a normal Sylow p-subgroup.*

One may use Dobson's theorem to prove an analogue to Theorem 5.2.3 for uniform hypergraphs of prime power order. The author poses the following problem.

Problem 5.4.3 *Characterize the structure of the vertex transitive self-complementary k-uniform hypergraphs of prime power order.*

In the case where $n = p^r \equiv 1 \pmod{2^{\ell+1}}$ for the largest element ℓ in the support of the binary representation of k, Theorem 3.1.4 implies that a self-complementary k-hypergraph X of order n cannot be 2-subset-regular, and hence cannot be doubly transitive. Therefore, if the automorphism group of X contains a cycle of length p^r, then it contains a normal Sylow p-subgroup. Examining the structure of such groups may lead to a partial solution to Problem 5.4.3.

Part II

Self-complementary nonuniform hypergraphs

Chapter 6

Introduction

6.1 Definitions

For a positive integer $n \geq 2$ and a nonempty subset K of $\{1, 2, \ldots, n-1\}$, a *K-hypergraph* (V, E) of order n is a hypergraph with vertex set V and edge set

$$E = \bigcup_{k \in K} E_k, \quad \text{where } E_k \subseteq V^{(k)} \text{ for all } k \in K.$$

The *complement* X^C of the K-hypergraph $X = (V, E)$ is the K-hypergraph with vertex set $V(X^C) = V$ and edge set $E(X^C) = \bigcup_{k \in K} V^{(k)} \setminus E$. An *isomorphism* between two K-hypergraphs $X = (V, E)$ and $Y = (W, F)$ is a bijection from V to W which induces a bijection from E to F. If such a bijection exists, we say that X and Y are *isomorphic*. The K-hypergraph X is called *self-complementary* if X and X^C are isomorphic. An isomorphism from a self-complementary K-hypergraph X to its complement X^C is called an *antimorphism* of X, and as usual we denote the set of antimorphisms of X by $\mathrm{Ant}(X)$. For each k in the rank set K of a K-hypergraph X, let X_k denote the subhypergraph of X induced by the edges of X of rank k. Note that if X is self-complementary, then the k-hypergraph X_k is self-complementary for all $k \in K$. Since any bijection maps edges of size k onto edges of size k, a permutation

θ is an antimorphism of X if and only if θ is an antimorphism of X_k for all $k \in K$. Thus $\mathrm{Ant}(X) = \bigcap_{k\in K} \mathrm{Ant}(X_k)$. Hence we have the following characterization of self-complementary K-hypergraphs.

Proposition 6.1.1 *A K-hypergraph $X = (V, E)$ is self-complementary if and only if both of the following conditions hold.*

(1) The subhypergraph X_k is a self-complementary k-hypergraph for all $k \in K$.

(2) The self-complementary subhypergraphs X_k, for all $k \in K$, share a common antimorphism θ. That is, $\bigcap_{k\in K} Ant(X_k) \neq \emptyset$.

A K-hypergraph $X = (V, E)$ is *t-subset-regular* if, for all $k \in K$, the subhypergraph $X_k = (V, E_k)$ induced by the edges of size k is t-subset-regular. An *automorphism* of a K-hypergraph X is an isomorphism from X to X, and as usual we denote the group of automorphisms of a K-hypergraph X by $\mathrm{Aut}(X)$. A K-hypergraph $X = (V, E)$ is *t-fold-transitive*, or simply *t-transitive*, if $\mathrm{Aut}(X)$ acts transitively on the set of ordered t-tuples of pairwise distinct elements of V. A 1-transitive K-hypergraph is called *vertex transitive*, and a 2-transitive K-hypergraph is called *doubly transitive*. In the language of design theory, the t-subset-regular self-complementary K-hypergraphs correspond to large sets of two isomorphic t-wise balanced designs, or t-partitions, in which the block sizes lie in the set K.

6.2 History and layout of part II

Szymański first introduced the notion of a self-complementary (non-uniform) hypergraph of order n in 2006 [31]. He defined it to be a self-complementary K-hypergraph for $K = \{1, 2, \ldots, n-1\}$.

The following theorem provides necessary and sufficient conditions on the order of a self-complementary K-hypergraph for $K = \{1, 2, \ldots, n-1\}$. The result was first conjectured by Szymański [31], who verified the result by computer for $n \leq 1000$. The theorem was then proved by Zwonek [39].

Theorem 6.2.1 **[39]** *Let* $K = \{1, 2, \ldots, n-1\}$*. There exists a self-complementary* K*-hypergraph of order* n *if and only if* $n = 2^\ell$ *for some positive integer* ℓ*.*

Actually, Zwonek proved the following stronger result.

Theorem 6.2.2 **[39]**

1. *Let* $K = \{1, 2, \ldots, n-1\}$*. If* X *is a self-complementary* K*-hypergraph of order* n*, then* $n = 2^\ell$ *for some positive integer* ℓ*.*

2. *If* $|V| = 2^\ell$*, then a permutation* $\theta \in Sym(V)$ *is a* k*-complementing permutation for all* $k \in \{1, 2, \ldots, 2^\ell - 1\}$ *if and only if* θ *is a cycle of length* 2^ℓ*.*

Observe that a self-complementary k-hypergraph is a self-complementary K-hypergraph for $K = \{k\}$. Hence the concept of a self-complementary K-hypergraph is a generalization of the previous concepts of self-complementary k-hypergraphs and Szymański's self-complementary non-uniform hypergraphs, as these structures are the two extreme cases of self-complementary K-hypergraphs.

In Part II of the thesis, we will examine the possible orders of self-complementary K-hypergraphs for various sets of positive integers K. The results in Part II rely on the necessary and sufficient conditions on the orders of self-complementary k-hypergraphs obtained in Part I of the thesis.

Part II is broken up into three chapters. In Chapter 7, we derive some necessary conditions on the order of self-complementary K-hypergraphs, and show that these conditions are sufficient in certain cases. In Chapters 8 and 9, we give a similar analysis of the orders of t-subset-regular self-complementary K-hypergraphs and t-fold-transitive self-complementary K-hypergraphs, respectively.

Chapter 7

Self-complementary K-hypergraphs

7.1 Necessary conditions on order

In this section, we obtain some necessary conditions on the order of a self-complementary K-hypergraph for certain sets K of positive integers.

Theorem 7.1.1 *Let $n \geq 2$ be an integer, and let $K \subseteq \{1, 2, \ldots, n-1\}$, $K \neq \emptyset$. Suppose that there exists a self-complementary K-hypergraph with n vertices.*

(1) If K contains a nonempty subset

$$L = \{2^\ell + 1, 2^{\ell+1} + 1, \ldots, 2^{\ell+r} + 1\}$$

for some integers ℓ, r with $\ell \geq 1$ and $r \geq 0$, then n is even or

$$n_{[2^{\ell+r+1}]} \in \{0, 1, \ldots, 2^\ell\}.$$

(2) If K contains a nonempty subset

$$M = \{2^\ell, 2^{\ell+1}, \ldots, 2^{\ell+r}\}$$

for some integers ℓ, r with $\ell \geq 1$ and $r \geq 0$, then

$$n_{[2^{\ell+r+1}]} \in \{0, 1, \ldots, 2^\ell - 1\}.$$

Proof: We prove the results by induction on r.

When $r = 0$, both results (1) and (2) follow directly from Corollary 2.3.3. Now suppose that r is a positive integer, and assume that both results (1) and (2) hold for $r-1$. Suppose that X is a self-complementary K-hypergraph with n vertices.

(1) If K contains a nonempty subset

$$L = \{2^\ell + 1, 2^{\ell+1} + 1, \ldots, 2^{\ell+r} + 1\}$$

for some positive integer ℓ, then the self-complementary subhypergraph of X induced by edges with sizes in $\hat{L} = L \setminus \{2^{\ell+r} + 1\}$ has order n and satisfies the hypothesis of condition (1) for $r-1$. Hence by the induction hypothesis,

$$n \text{ is even or } n_{[2^{\ell+(r-1)+1}]} \in \{0, 1, \ldots, 2^\ell\}. \tag{7.1.1}$$

Also, the subhypergraph of X induced by the edges of size $2^{\ell+r} + 1$ is a self-complementary $(2^{\ell+r} + 1)$-hypergraph of order n, and so Corollary 2.3.3 implies that

$$n \text{ is even or } n_{[2^{\ell+r+1}]} \in \{0, 1, \ldots, 2^{\ell+r}\}. \tag{7.1.2}$$

If n is not even, then (7.1.2) guarantees that $n_{[2^{\ell+r+1}]} < 2^{\ell+r}$, which implies that $n_{[2^{\ell+r+1}]} = n_{[2^{\ell+r}]}$, and so (7.1.2) guarantees that $n_{[2^{\ell+r+1}]} \in \{0, 1, \ldots, 2^\ell\}$. Hence either n is even or $n_{[2^{\ell+r+1}]} \in \{0, 1, \ldots, 2^\ell\}$, and so (1) holds for r. Thus by the principle of mathematical induction, (1) holds for all $r \geq 0$.

(2) If K contains a nonempty subset $M = \{2^\ell, 2^{\ell+1}, \ldots, 2^{\ell+r}\}$ for some positive integer ℓ, then the self-complementary subhypergraph of X induced by the edges with sizes in $\hat{M} = M \setminus \{2^{\ell+r}\}$ has order n and satisfies the hypothesis of condition (2) for $r-1$. Hence by the induction hypothesis,

$$n_{[2^{\ell+(r-1)+1}]} \in \{0, 1, \ldots, 2^\ell - 1\}. \tag{7.1.3}$$

Also, the subhypergraph of X induced by the edges of size $2^{\ell+r}$ is a self-complementary $2^{\ell+r}$-hypergraph of order n, and so Corollary 2.3.3 implies that

$$n_{[2^{\ell+r+1}]} \in \{0, 1, \ldots, 2^{\ell+r} - 1\}. \qquad (7.1.4)$$

Now conditions (7.1.3) and (7.1.4) together imply that the result in (2) holds for r. Thus by the principle of mathematical induction, (2) holds for all $r \geq 0$. ∎

7.2 Sufficient conditions on order

In this section, we use the characterization of the lengths of the orbits of a k-complementing permutation in Theorem 2.2.5 to obtain some sufficient conditions on the order of self-complementary K-hypergraphs, for various sets K.

In the first theorem, we show that the necessary conditions in Theorem 7.1.1 are also sufficient in the cases where $L = K$ or $M = K$.

Theorem 7.2.1 *Let $n \geq 2$ be an integer, and let L and M be nonempty subsets of $\{1, 2, \ldots, n-1\}$.*

(1) If $L = \{2^\ell + 1, 2^{\ell+1} + 1, \ldots, 2^{\ell+r} + 1\}$ for some integers ℓ, r with $\ell \geq 1$ and $r \geq 0$, then there exists a self-complementary L-hypergraph of order n whenever n is even or $n_{[2^{\ell+r+1}]} \in \{0, 1, \ldots, 2^\ell\}$.

(2) If $M = \{2^\ell, 2^{\ell+1}, \ldots, 2^{\ell+r}\}$ for some integers ℓ, r with $\ell \geq 1$ and $r \geq 0$, then there exists a self-complementary M-hypergraph of order n whenever $n_{[2^{\ell+r+1}]} \in \{0, 1, \ldots, 2^\ell - 1\}$.

Proof: It suffices to show the following:

(i) If n is even, then there exists a permutation in $Sym(n)$ that is k-complementing for all $k \in L$.

(ii) If $n_{[2^{\ell+r+1}]} \in \{0, 1, \ldots, 2^\ell\}$, then

(a) there exists a permutation in $Sym(n)$ that is k-complementing for all $k \in L$, and

(b) if $n_{[2^{\ell+r+1}]} < 2^\ell$, then there exists a permutation in $Sym(n)$ that is k-complementing for all $k \in M$.

(i) Suppose that n is even, say $n = 2m$ for a positive integer m, and let $\theta \in Sym(n)$ such that θ has m orbits of length 2. Fix $k \in L$, and let b be the binary representation of k. We will use Theorem 2.2.5 to show that θ is a k-complementing permutation. Let $V = \{1, 2, \ldots, n\}$, let $A = \emptyset$ and let $B = V$. Then $A \cap B = \emptyset$, $A \cup B = V$, and A and B are both equal to unions of orbits of θ. Moreover, since $k \in L$, we have $0 \in \text{supp}(b)$. Now $|A| = 0 < k_{[2^0+1]}$, and it is vacuously true that every cycle of $\theta|_A$ has length 2^r for some integer $r < 0$, so condition (I) of Theorem 2.2.5 holds for A with $\ell = 0$. Also, every cycle of $\theta|_B$ has length $2 = 2^1$, and so as $1 > 0$, condition (II) of Theorem 2.2.5 holds for B with $\ell = 0$. Hence Theorem 2.2.5 guarantees that θ is a k-complementing permutation, and since k was an arbitrary element of L, θ is a k-complementing permutation for all $k \in L$. Hence (i) holds.

(ii) Now suppose that $n \equiv j \pmod{2^{\ell+r+1}}$ for some $j \in \{0, 1, \ldots, 2^\ell\}$, say $n = m2^{\ell+r+1} + j$. Let $\sigma \in Sym(n)$ such that σ has m orbits of length $2^{\ell+r+1}$ and j fixed points. Fix $k \in L \cup M$, and let b be the binary representation of k. Now $k = 2^{\ell+i}$ or $k = 2^{\ell+i} + 1$ for some $i \in \{0, 1, \ldots, r\}$, and so $\ell + i \in \text{supp}(b)$. To prove (a) and (b), we will use Theorem 2.2.5 to show that σ is a k-complementing permutation for $k = 2^{\ell+i} + 1$, and for $k = 2^{\ell+i} + 1$ when $j < 2^\ell$. Let $V = \{1, 2, \ldots, n\}$, let A be the set of j fixed points of σ, and let B be the set of elements of V which lie in a cycle of σ of length $2^{\ell+r+1}$. Then A and B are both equal to unions of orbits of σ, and $A \cup B = V$. Since $2^{\ell+r+1} > 1$, we also have $A \cap B = \emptyset$.

First we will show that conditions (I) of Theorem 2.2.5 holds for σ with the set A and the element $\ell + i \in \text{supp}(b)$, under the conditions of (a) and (b).

(a) $k \in L$ *and* $j \leq 2^\ell$. Then $k = 2^{\ell+i} + 1$. In this case $|A| = j \leq 2^\ell < 2^{\ell+i} + 1 = k_{[2^{\ell+i+1}]}$, and so condition (I) holds for A with the element $\ell + i \in \text{supp}(b)$.

(b) $k \in M$ *and* $j < 2^\ell$. Then $k = 2^{\ell+i}$. Hence $|A| = j < 2^\ell \leq 2^{\ell+i} = k_{[2^{\ell+i+1}]}$, and so condition (I) holds for A with the element $\ell + i \in \text{supp}(b)$ in this case also.

Now observe that, in both cases (a) and (b) above, every cycle of $\sigma\mid_B$ has length $2^{\ell+r+1}$, and the fact that $r \geq i$ guarantees that $\ell + r + 1 > \ell + i$. Hence condition (II) of Theorem 2.2.5 holds for σ with the set B and the element $\ell + i \in \text{supp}(b)$, in both cases (a) and (b). Thus Theorem 2.2.5 guarantees that σ is a k-complementing permutation in both cases. It follows that (ii)(a) and (ii)(b) both hold. This completes the proof. ∎

Let $\mathbb{N}_2$ denote the set of positive integers which are sums of consecutive powers of 2. That is,

$$\mathbb{N}_2 = \{1, 2, 2+1, 2^2, 2^2+2, 2^2+2+1, 2^3, 2^3+2^2, 2^3+2^2+2, 2^3+2^2+2+1, \ldots\}.$$

In the next theorem, we will use Corollary 2.3.4 to obtain necessary conditions on the order of self-complementary K-hypergraphs in the case where K contains a set $\hat{K}$ of consecutive elements from $\mathbb{N}_2$, and we show that these necessary conditions are sufficient when $\hat{K} = K$.

Theorem 7.2.2 *Let K be a set of positive integers, let $k_{\max} = \max\{k : k \in K\}$, and let n be an integer such that $n \geq k_{\max} + 1$. Suppose that K contains a nonempty* *subset* $\hat{K}$ *of consecutive elements in* $\mathbb{N}_2$*. Let* $k^* = \max\{k : k \in \hat{K}\}$ *and let* $k_* = \min\{k : k \in \hat{K}\}$*. Let* b^* *and* b_* *be the binary representation of* k^* *and* k_**, respectively. Let* ℓ^* *and* ℓ_* *denote the largest elements in supp*(b^*) *and supp*(b_*)*, respectively.*

(1) If there exists a self-complementary K*-hypergraph of order* n*, then* $n_{[2^{\ell^*}+1]} < k_*$*.*

(2) If $K = \hat{K}$*, then there exists a self-complementary* K*-hypergraph of order* n *if and only if* $n_{[2^{\ell^*}+1]} < k_*$*.*

Proof:

(1) Suppose that there exists a self-complementary K-hypergraph X of order n. We will show that $n_{[2^{\ell^*}+1]} < k_*$ by induction on $\ell^* - \ell_*$. Note that by the definition of $\mathbb{N}_2$, since $k^* \geq k_*$, we have $\ell^* \geq \ell_*$.

Base Step: If $\ell^* - \ell_* = 0$, then

$$\hat{K} = \left\{ \sum_{i=c}^{\ell^*} 2^i, \sum_{i=c-1}^{\ell^*} 2^i, \ldots, \sum_{i=c-d}^{\ell^*} 2^i \right\}$$

for some nonnegative integers c, d such that $0 \leq c - d \leq c \leq \ell^*$. Now for each $k \in \hat{K}$, the self-complementary subhypergraph X_k of X induced by the edges with size k has order n, and hence Corollary 2.3.4 implies that $n_{[2^{\ell^*}+1]} < k$ for all $k \in \hat{K}$. Thus $n_{[2^{\ell^*}+1]} < \min\{k : k \in \hat{K}\} = k_*$, as required.

Induction Step: Let r be an integer, $r \geq 1$.

- *Induction Hypothesis:* If K contains a nonempty subset $\hat{M}$ of consecutive elements in $\mathbb{N}_2$, $m^* = \max\{k : k \in \hat{M}\}$, $m_* = \min\{k : k \in \hat{M}\}$, a^* and a_* denote the largest elements in the supports of the binary representations of m^* and m_*, respectively, and $a^* - a_* = r - 1$, then $n_{[2^{a^*}+1]} < m_*$.

Now suppose that $\ell^* - \ell_* = r$. Let b_k denote the binary representation of a positive integer k. Let

$$M = \{k \in \hat{K} : \max\{i : i \in \text{supp}(b_k)\} = \ell^*\},$$

and let $\hat{M} = \hat{K} \setminus M$. Let $m^* = \max\{k : k \in \hat{M}\}$, let $m_* = \min\{k : k \in \hat{M}\} = k_*$, and let a^* and a_* denote the largest elements in the supports of the binary representations of m^* and m_*, respectively. Then $a^* = \ell^* - 1$ and $a_* = \ell_*$, and so $a^* - a_* = r - 1$. Hence by the induction hypothesis, we have $n_{[2^{a^*+1}]} < m_*$. Since $m_* = k_*$ and $a^* = \ell^* - 1$, this implies that

$$n_{[2^{\ell^*}]} < k_*. \tag{7.2.1}$$

Now by definition, we have

$$M = \left\{2^{\ell^*}, 2^{\ell^*} + 2^{\ell^*-1}, \ldots, \sum_{i=\ell^*-c}^{\ell^*} 2^i\right\}$$

for some integer c such that $0 \leq c \leq \ell^*$. Hence $2^{\ell^*} \in M \subseteq K$, and so by Theorem 7.1.1(2) we obtain $n_{[2^{\ell^*+1}]} < 2^{\ell^*}$. But this implies that $n_{[2^{\ell^*+1}]} = n_{[2^{\ell^*}]}$. Putting this together with (7.2.1), we obtain

$$n_{[2^{\ell^*+1}]} < k_*,$$

as required. ∎

(2) Suppose that $K = \hat{K}$ and there exists a self-complementary K-hypergraph of order n. Then $n_{[2^{\ell^*+1}]} < k_*$ by part (1).

Conversely, suppose that $K = \hat{K}$, and let n be an integer such that $n_{[2^{\ell^*+1}]} < k_*$. Then $n = M2^{\ell^*+1} + j$ for a positive integer M and an integer $j \in \{0, 1, \ldots, k_* - 1\}$. We will show that there exists a self-complementary K-hypergraph of order n.

It suffices to show that there exists a permutation in $Sym(n)$ that is k-complementing for all $k \in K$.

Let $\theta \in Sym(n)$ be a permutation whose disjoint cycle decomposition has M cycles of length 2^{ℓ^*+1} and j cycles of length 1 (j fixed points). Let k be an arbitrary element of K and let b be the binary representation of k. Since $K = \hat{K}$, it follows that $k \in \mathbb{N}_2$. Let $\ell = \max\{i : i \in \text{supp}(b)\}$. By definition of ℓ^*, we have $\ell \leq \ell^*$. Moreover, by definition of k_*, we have $j < k_* \leq k$, and so $j < k$. Let A be the set of fixed points of θ, and let B be the set of points in $\{1, 2, \ldots, n\}$ which lie in a cycle of θ of length 2^{ℓ^*+1}. Then certainly $A \cap B = \emptyset$ and $A \cup B = domain(\theta)$. Also, the sets A and B are both equal to unions of orbits of θ. Moreover, every cycle of B has length 2^{ℓ^*+1} and $\ell^* + 1 \geq \ell + 1 > \ell$, and so θ satisfies condition (II) of Theorem 2.2.5. Also, we have that $|A| = j < k$ and every cycle of $\theta\mid_A$ has length 2^0. Hence if $\ell > 0$, then θ satisfies condition (I) of Theorem 2.2.5. On the other hand, if $\ell = 0$, then $k = 1$, and so the fact that $|A| < k$ implies that $|A| = 0$, and hence θ satisfies condition (I) of Theorem 2.2.5 in this case also. Thus Theorem 2.2.5 guarantees that θ is a k-complementing permutation. Since k was arbitrary, we conclude that θ is a k-complementing permutation for every $k \in K$. Hence there exists a self-complementary K-hypergraph of order n. ∎

Chapter 8

Regular self-complementary K-hypergraphs

8.1 Necessary conditions on order

In this section we determine necessary conditions on the order n of a t-subset-regular self-complementary K-hypergraph for various subsets K of $\{1, 2, \ldots, n-1\}$.

Throughout this section, for a positive integer k, let b_k denote the binary representation of k.

Theorem 8.1.1 *Let $n \geq 2$ be an integer, let $K \subseteq \{1, 2, \ldots, n-1\}$, $K \neq \phi$, and let t be an integer such that $1 \leq t < \min\{k : k \in K\}$. Suppose that there exists a t-subset-regular self-complementary K-hypergraph with n vertices. Then the following conditions hold:*

(1) If $K = \{k\}$ for some $k \in \{1, 2, \ldots, n-1\}$, then there exists an integer $\ell \in supp(b_k)$ such that

$$n_{[2^{\ell+1}]} \in \{t, t+1, \ldots, k_{[2^{\ell+1}]} - 1\}.$$

(2) $K \neq \{1, 2, \ldots, n-1\}$.

(3) If K contains a nonempty set

$$L = \{2^\ell, 2^\ell + 1, 2^{\ell+1}, 2^{\ell+1} + 1, \ldots, 2^{\ell+r}, 2^{\ell+r} + 1\} \setminus S$$

for some $S \subseteq \{2^\ell, 2^{\ell+r} + 1\}$ and some integers ℓ, r with $\ell \geq 1$ and $r \geq 0$, then

$$n_{[2^{\ell+r+1}]} \in \{t, t+1, \ldots, L_{\min} - 1\},$$

where $L_{\min} = \min\{k : k \in L\}$.

(4) Suppose that K contains a nonempty subset $\hat{K}$ of consecutive elements in $\mathbb{N}_2$. Let $k^* = \max\{k : k \in \hat{K}\}$ and let $k_* = \min\{k : k \in \hat{K}\}$. Let b^* and b_* be the binary representation of k^* and k_*, respectively. Let ℓ^* and ℓ_* denote the largest elements in supp(b^*) and supp(b_*), respectively. Then $n_{[2^{\ell^*+1}]} \in \{t, t+1, \ldots, k_* - 1\}$.

Proof:

(1) If $K = \{k\}$ for some $k \in \{1, 2, \ldots, n-1\}$, then X is a t-subset-regular self-complementary k-hypergraph with n vertices, and so the result follows directly from Theorem 3.1.4.

(2) Suppose, for the sake of contradiction, that $K = \{1, 2, \ldots, n-1\}$. Then the subhypergraph $X_k = (V, E \cap V^{(k)})$ is a t-subset-regular self-complementary k-uniform hypergraph for all $k \in \{1, \ldots, n-1\}$. But then X_1 is a 1-subset-regular 1-hypergraph, which is impossible.

(3) Fix a positive integer ℓ. We prove the result by induction on r.

If $r = 0$, then $L = \{2^\ell, 2^\ell + 1\} \setminus S$ for some $S \subseteq \{2^\ell, 2^\ell + 1\}$. Since $L \neq \phi$ by assumption, it follows that L must be a nonempty subset of $\{2^\ell, 2^\ell + 1\}$. If $|L| = 1$, then $X_k = (V, E_k)$ is a t-subset-regular self-complementary k-hypergraph for $k = 2^\ell$ or $k = 2^\ell + 1$. Thus in this case the result follows directly from Corollary 3.1.5. If $|L| = 2$, then $L = \{2^\ell, 2^\ell + 1\}$, and so $X_k = (V, E_k)$ is a self-complementary t-subset-regular k-hypergraph for both $k = 2^\ell$ and $k = 2^\ell + 1$. Hence Corollary 3.1.5 implies that

$$\begin{aligned} n_{[2^{\ell+1}]} &\in \{t, t+1, \ldots, 2^\ell - 1\} \cap \{t, t+1, \ldots, 2^\ell\} \\ &= \{t, t+1, \ldots, 2^\ell - 1\} \\ &= \{t, t+1, \ldots, L_{\min} - 1\}. \end{aligned}$$

Hence the result is true for $r = 0$.

Now suppose that r is a positive integer.

- **Induction Hypothesis:** If there exists a t-subset-regular self-complementary $\hat{K}$-hypergraph, where $\hat{K}$ contains a nonempty set

  $$\hat{L} = \{2^\ell, 2^\ell + 1, 2^{\ell+1}, 2^{\ell+1} + 1, \ldots, 2^{\ell+r-1}, 2^{\ell+r-1} + 1\} \setminus \hat{S}$$

  for some $\hat{S} \subseteq \{2^\ell, 2^{\ell+r-1} + 1\}$, then

  $$n_{[2^{\ell+r}]} \in \{t, t+1, \ldots, \hat{L}_{\min} - 1\},$$

  where $\hat{L}_{\min} = \min\{k : k \in \hat{L}\}$.

Suppose that K contains

$$L = \{2^\ell, 2^\ell + 1, 2^{\ell+1}, 2^{\ell+1} + 1, \ldots, 2^{\ell+r}, 2^{\ell+r} + 1\} \setminus S$$

for some $S \subseteq \{2^\ell, 2^{\ell+r} + 1\}$. Let

$$\hat{L} = \{2^\ell, 2^\ell + 1, 2^{\ell+1}, 2^{\ell+1} + 1, \ldots, 2^{\ell+r-1} + 1\} \setminus S.$$

Then the subhypergraph of X induced by the edges in $\cup_{k \in \hat{L}} E_k$ is a t-subset-regular self-complementary $\hat{L}$-hypergraph, and so by the induction hypothesis,

$$n_{[2^{\ell+r}]} \in \{t, t+1, \ldots, \hat{L}_{\min} - 1\} = \{t, t+1, \ldots, L_{\min} - 1\} \tag{8.1.1}$$

since $\hat{L}_{\min} = \min\{k : k \in \hat{L}\} = \min\{k : k \in L\} = L_{\min}$. Moreover, the subhypergraph of X induced by the edges in $\cup_{k \in L^*} E_k$ is a t-subset-regular self-complementary L^*-hypergraph for $L^* = \{2^{\ell+r}\}$ or $L^* = \{2^{\ell+r}, 2^{\ell+r}+1\}$. In either case, Corollary 3.1.5 implies that

$$n_{[2^{\ell+r+1}]} \in \{t, t+1, \ldots, 2^{\ell+r} - 1\}. \tag{8.1.2}$$

Thus n satisfies both (8.1.1) and (8.1.2). Both of these conditions hold if and only if

$$n_{[2^{\ell+r+1}]} \in \{t, t+1, \ldots, L_{\min} - 1\},$$

and thus the result is true for r. By the principle of mathematical induction, we conclude that the result holds for all nonnegative integers r, for any fixed positive integer ℓ.

(4) Suppose that there exists a self-complementary K-hypergraph of order n. We will show that $n_{[2^{\ell^*+1}]} \in \{t, t+1, \ldots, k_* - 1\}$ by induction on $\ell^* - \ell_*$.

Base Step: If $\ell^* - \ell_* = 0$, then

$$\hat{K} = \left\{ \sum_{i=c}^{\ell^*} 2^i, \sum_{i=c-1}^{\ell^*} 2^i, \ldots, \sum_{i=c-d}^{\ell^*} 2^i \right\}$$

for some nonnegative integers c, d such that $0 \leq c - d \leq c \leq \ell^*$. Now for each $k \in \hat{K}$, the t-subset-regular self-complementary subhypergraph X_k of X induced by the edges with size k has order n, and hence Corollary 3.1.6 implies that $n_{[2^{\ell^*+1}]} \in \{t, t+1, \ldots, k-1\}$ for all $k \in \hat{K}$. Since $k_* \in \hat{K}$, we have $n_{[2^{\ell^*+1}]} \in \{t, t+1, \ldots, k_* - 1\}$, as required.

Induction Step: Let r be an integer, $r \geq 1$.

- *Induction Hypothesis:* If K contains a nonempty subset $\hat{M}$ of consecutive elements in $\mathbb{N}_2$, $m^* = \max\{k : k \in \hat{M}\}$, $m_* = \min\{k : k \in \hat{M}\}$, a^* and

a_* denote the largest elements in the supports of the binary representations of m^* and m_*, respectively, and $a^* - a_* = r - 1$, then $n_{[2^{a^*}+1]} \in \{t, t+1, \ldots, m_* - 1\}$.

Now suppose that $\ell^* - \ell_* = r$. Let

$$M = \{k \in \hat{K} : \max\{i : i \in \mathrm{supp}(b_k)\} = \ell^*\},$$

and let $\hat{M} = \hat{K} \setminus M$. Let $m^* = \max\{k : k \in \hat{M}\}$, let $m_* = \min\{k : k \in \hat{M}\} = k_*$, and let a^* and a_* denote the largest elements in the supports of the binary representations of m^* and m_*, respectively. Then $a^* = \ell^* - 1$ and $a_* = \ell_*$, and so $a^* - a_* = r - 1$. Hence by the induction hypothesis, we have $n_{[2^{a^*}+1]} \in \{t, t+1, \ldots, m_* - 1\}$. Since $m_* = k_*$ and $a^* = \ell^* - 1$, this implies that

$$n_{[2^{\ell^*}]} \in \{t, t+1, \ldots, k_* - 1\}. \tag{8.1.3}$$

Now by definition, we have

$$M = \left\{ 2^{\ell^*}, 2^{\ell^*} + 2^{\ell^*-1}, \ldots, \sum_{i=\ell^*-c}^{\ell^*} 2^i \right\}$$

for some integer c such that $0 \leq c \leq \ell^*$. Hence $2^{\ell^*} \in M \subseteq K$. Since the subhypergraph $X_{2^{\ell^*}}$ is t-subset-regular, Corollary 3.1.5 implies that $n_{[2^{\ell^*}+1]} \in \{t, t+1, \ldots, 2^{\ell^*} - 1\}$. But this implies that $n_{[2^{\ell^*}+1]} = n_{[2^{\ell^*}]}$. Putting this together with (8.1.3), we obtain

$$n_{[2^{\ell^*}+1]} \in \{t, t+1, \ldots, k_* - 1\},$$

as required. ∎

8.2 Constructions

In this section, we construct some 1-subset-regular self-complementary K-hypergraphs, and prove that that necessary conditions established in Section 8.1 are sufficient for certain sets K.

Construction 8.2.1 Let K be a set of positive integers such that $k_{\min} = \min\{k : k \in K\} \geq 2$. Let $k_{\max} = \max\{k : k \in K\}$, let $b = (b_m, b_{m-1}, \ldots, b_1, b_0)$ be the binary representation of $k_{\max}$, and let $\ell = \max\{i : b_i = 1\}$. Let $s \in \{1, 2, \ldots, k_{\min} - 1\}$, let m be a positive integer, let $S = \{\infty_1, \ldots, \infty_s\}$ such that $S \cap \mathbb{Z}_{m2^{\ell+1}} = \emptyset$, and let $V = S \cup \mathbb{Z}_{m2^{\ell+1}}$. For each $k \in K$, let X_k be a 1-subset-regular self-complementary k-hypergraph on V with antimorphism

$$\theta = (\infty_1)(\infty_2)\cdots(\infty_s)\prod_{j=0}^{m-1}(j2^{\ell+1}, j2^{\ell+1}+1, \ldots, (j+1)2^{\ell+1}-1)$$

given by Lemma 3.2.5 (for $a = \ell + 1$).

Define X to be the K-hypergraph with vertex set V and edge set $E = \cup_{k\in K} E(X_k)$.

Lemma 8.2.2 *The K-hypergraph X of Construction 8.2.1 is 1-subset-regular and self-complementary.*

Proof: The definition of ℓ guarantees that $k_{\max[2^{\ell+1}]} = k_{\max}$. Hence for each $k \in K$ we have $k_{[2^{\ell+1}]} = k$, and thus $1 \leq s < k_{[2^{\ell+1}]} = k$. Also, since $k_{\min} \geq 2$, we have $k_{\max} \geq 2$, and so we are guaranteed that $\ell + 1 \geq 2$. Hence the k-hypergraphs X_k given by Lemma 3.2.5 exist for all $k \in K$. Now the subhypergraph of X induced by the edges of rank k is $X_k = (V, E(X_k))$, which is 1-subset-regular, and so X is 1-subset-regular. Moreover, since each subhypergraph X_k is self-complementary and $\theta \in \cap_{k\in K}\mathrm{Ant}(X_k)$, Proposition 6.1.1 implies that X is self-complementary. ∎

Construction 8.2.1 and Lemma 8.2.2 together imply the following sufficient conditions on the order of a 1-subset-regular self-complementary K-hypergraph, for any set of positive integers K.

Theorem 8.2.3 *Let K be a set of positive integers such that $k_{\min} = \min\{k : k \in K\} \geq 2$. Let $k_{\max} = \max\{k : k \in K\}$, let b be the binary representation of $k_{\max}$, and let ℓ be the largest element in supp(b). If n is an integer such that $n \geq k_{\max} + 1$ and $n_{[2^{\ell+1}]} \in \{1, 2, \ldots, k_{\min} - 1\}$, then there exists a 1-subset-regular self-complementary K-hypergraph of order n.* ∎

Theorem 8.2.3 implies that the necessary conditions on the order of a 1-subset-regular self-complementary K-hypergraph given by Theorem 8.1.1(3) and (4) are sufficient for certain sets K.

Theorem 8.2.4 *Let K be a set of positive integers such that $k_{\min} = \min\{k : k \in K\} \geq 2$. Let $k_{\max} = \max\{k : k \in K\}$, let b be the binary representation of $k_{\max}$, and let ℓ be the largest element in supp(b). If K satisfies one of the following conditions, then there exists a 1-subset-regular self-complementary K-hypergraph of order n if and only if $n \geq k_{\max} + 1$ and $n_{[2^{\ell+1}]} \in \{1, 2, \ldots, k_{\min} - 1\}$.*

(1) $K = \{2^i, 2^i + 1, 2^{i+1}, 2^{i+1} + 1, \ldots, 2^{i+r}, 2^{i+r} + 1\} \setminus S$ for some $S \subseteq \{2^i, 2^{i+r} + 1\}$ and some integers i, r with $i \geq 1$ and $r \geq 0$.

(2) K is a set of consecutive elements from $\mathbb{N}_2$.

(3) K is a set of consecutive positive integers such that $k_{\min} \in \mathbb{N}_2$.

Proof: In all three cases, the sufficiency of the conditions on n follows from Theorem 8.2.3. In cases (1) and (2), the necessity of the conditions on n follow from parts (3) and (4) of Theorem 8.1.1, respectively.

It remains to show that the condition $n_{[2^{\ell+1}]} \in \{1, 2, \ldots, k_{\min} - 1\}$ is necessary in case (3). Suppose that K is a set of consecutive positive integers such that $k_{\min} \in \mathbb{N}_2$, and there exists a 1-subset-regular self-complementary K-hypergraph of order n. The set K contains a nonempty subset $\hat{K} = K \cap \mathbb{N}_2$ of consecutive elements of $\mathbb{N}_2$. Let $k_* = \min\{k : k \in \hat{K}\}$, let $k^* = \max\{k : k \in \hat{K}\}$, let b^* be the binary representation of k^*, and let ℓ^* denote the largest element in $\text{supp}(b^*)$. Then Theorem 8.1.1(4) implies that $n_{[2^{\ell^*+1}]} \in \{1, 2, \ldots, k_* - 1\}$. Since the largest element in $\text{supp}(b^*)$ must equal the largest element of $\text{supp}(b)$, we have $\ell^* = \ell$. Moreover, since $k_{\min} \in \mathbb{N}_2$, we also have $k_* = k_{\min}$. Hence $n_{[2^{\ell+1}]} \in \{1, 2, \ldots, k_{\min} - 1\}$, as required. ∎

In the next construction, we use Lemma 3.2.5 to find some sufficient conditions on the order of a 1-subset-regular self-complementary K-hypergraph for any nonempty set of positive integers K.

Construction 8.2.5 Let $a \geq 2$ be an integer, let K be a set of positive integers, and let $\hat{k}_{\min} = \min\{k_{[2^a]} : k \in K\}$. Let $s \in \{1, 2, \ldots, \hat{k}_{\min} - 1\}$, let m be a positive integer, let $S = \{\infty_1, \ldots, \infty_s\}$ such that $S \cap \mathbb{Z}_{m2^{\ell+1}} = \emptyset$, and let $V = S \cup \mathbb{Z}_{m2^{\ell+1}}$. For each $k \in K$, let Y_k be the k-hypergraph on V given by Lemma 3.2.5.

Define Y to be the K-hypergraph with vertex set V and edge set $E = \cup_{k \in K} E(Y_k)$.

Lemma 8.2.6 *The K-hypergraph Y of Construction 8.2.5 is 1-subset-regular and self-complementary.*

Proof: First, since $a \geq 2$ and $1 \leq s < k_{[2^a]}$ for all $k \in K$, the hypergraphs Y_k given by Lemma 3.2.5 exist. Thus for each $k \in K$, the subhypergraph $Y_k = (V, E(Y_k))$ of Y induced by the edges of rank k is 1-subset-regular and self-complementary. Moreover, the permutation

$$\theta = (\infty_1)(\infty_2) \cdots (\infty_s) \prod_{j=0}^{m-1} (j2^{\ell+1}, j2^{\ell+1} + 1, \ldots, (j+1)2^{\ell+1} - 1)$$

is an antimorphism of Y_k, for all $k \in K$, and so $\theta \in \cap_{k\in K}\mathrm{Ant}(Y_k)$. It follows that Y is 1-subset-regular and self-complementary. ∎

Construction 8.2.5 and Lemma 8.2.6 together imply the following sufficient conditions on the order of a 1-subset-regular self-complementary K-hypergraph, for any nonempty set of positive integers K.

Theorem 8.2.7 *Let $a \geq 2$ be an integer, let K be a set of positive integers, let $\hat{k}_{\min} = \min\{k_{[2^a]} : k \in K\}$ and let $k_{\max} = \max\{k : k \in K\}$. If $n \geq k_{\max} + 1$ and $n_{[2^a]} \in \{1, 2, \ldots, \hat{k}_{\min} - 1\}$, then there exists a 1-subset-regular self-complementary K-hypergraph of order n.* ∎

Chapter 9

Transitive self-complementary K-hypergraphs

9.1 Necessary conditions on order

In this section we will determine necessary conditions on the order n of t-fold-transitive self-complementary K-hypergraphs for various subsets K of $\{1, 2, \ldots, n\}$.

Let $X = (V, E)$ be a K-hypergraph. Recall that, for each $k \in K$, the symbol X_k denotes the subhypergraph of X induced by the edges of X of size k. Since each permutation in $\mathrm{Aut}(X)$ maps edges of X_k onto edges of X_k, it follows that $\theta \in \mathrm{Aut}(X)$ if and only if $\theta \in \mathrm{Aut}(X_k)$ for all $k \in K$. Thus $\mathrm{Aut}(X) = \cap_{k \in K} \mathrm{Aut}(X_k)$, which implies that X_k inherits the transitivity properties of X, for all $k \in K$. Hence if X is t-fold-transitive, then X_k is t-fold-transitive for all $k \in K$.

Since t-fold-transitive k-hypergraphs are necessarily t-subset-regular, it follows that t-fold-transitive K-hypergraphs are also t-subset-regular. Hence we can use Theorem 8.1.1 to find basic necessary conditions on the orders of t-fold-transitive self-complementary K-hypergraphs for certain subsets K of the set $\{1, 2, \ldots, n-1\}$. However, the following result shows that the property of t-fold-transitivity implies even stronger necessary conditions on the order n of self-complementary K-hypergraphs than those given by Theorem 8.1.1(3) in the case where $n \equiv t \pmod{2^{\ell+r+1}}$.

Theorem 9.1.1 *Let K be a nonempty set of integers which contains a nonempty set L of integers of the form 2^ℓ or $2^\ell+1$ for a positive integer ℓ. Let $L_{\max} = \max\{k : k \in L\}$, let $m = \min\{i : 2^i > L_{\max}\}$, and let n be an integer such that $n \equiv t \pmod{2^m}$. Suppose that there exists a t-fold-transitive self-complementary K-hypergraph $X = (V, E)$ of order n. Then*

$$p^{(n-t+1)_{(p)}} \equiv 1 \pmod{2^m} \quad \textit{for every prime } p.$$

Proof: We know that $L_{\max} = 2^\ell$ or $2^\ell+1$ for some positive integer ℓ. Then $m = \ell+1$. Observe that the subhypergraph $X_{L_{\max}} = (V, E_{L_{\max}})$ is a t-fold-transitive self-complementary k-hypergraph of order $n \equiv t \pmod{2^{\ell+1}}$ for $k = 2^\ell$ or $k = 2^\ell + 1$. Thus Theorem 4.1.3 implies that

$$p^{(n-t+1)_{(p)}} \equiv 1 \pmod{2^{\ell+1}} \quad \text{for every prime } p.$$

Since $m = \ell + 1$, this completes the proof. ∎

9.2 Constructions

In this section, we present several constructions for vertex transitive self-complementary K-hypergraphs, and thus obtain some sufficient conditions on the order of these structures, for various sets K of positive integers.

9.2.1 Paley K-hypergraphs

We begin with a construction for vertex transitive self-complementary K-hypergraphs of prime power order, which is an extension of Construction 4.2.4 for the Paley k-

uniform hypergraphs $P_{q,k,r}$.

Recall that for a prime power q, an element $a \in \mathbb{F}_q^*$, and an element $b \in \mathbb{F}_q$, we define the mapping $\alpha_{a,b} : \mathbb{F}_q \to \mathbb{F}_q$ by $x^{\alpha_{a,b}} = ax + b$ for all $x \in \mathbb{F}_q$.

Construction 9.2.1 ***Paley* K*-hypergraph***

Let K be a set of positive integers such that $\min\{k : k \in K\} \geq 2$. For each $k \in K$, let

$$\ell_k = \max\{k_{(2)}, (k-1)_{(2)}\},$$

and let

$$\ell = \max\{\ell_k : k \in K\}.$$

Let q be a prime power such that $q \equiv 1 \pmod{2^{\ell+1}}$, and let r be a divisor of the integer $(q-1)/2^{\ell+1}$. Let $\mathbb{F}_q$ be the field of order q.

Define $P_{q,K,r}$ to be the K-hypergraph with vertex set

$$V(P_{q,K,r}) = \mathbb{F}_q$$

and edge set

$$E(P_{q,K,r}) = \bigcup_{k \in K} E(P_{q,k,r}),$$

in which $P_{q,k,r}$ is the Paley k-uniform hypergraph of Construction 4.2.4.

Lemma 9.2.2 *The Paley K-hypergraph $P_{q,K,r}$ defined in Construction 9.2.1 is a vertex transitive and self-complementary K-hypergraph.*

Proof: Let $k \in K$. Since $\ell \geq \ell_k$ and $q \equiv 1 \pmod{2^{\ell+1}}$, it follows that $q \equiv 1 \pmod{2^{\ell_k+1}}$. Also, since

$$\frac{q-1}{2^{\ell_k+1}} = \frac{q-1}{2^{\ell_k+1}} \frac{2^{\ell-\ell_k}}{2^{\ell-\ell_k}} = \frac{q-1}{2^{\ell+1}} 2^{\ell-\ell_k}$$

and r divides $(q-1)/2^{\ell+1}$, it follows that r also divides $(q-1)/2^{\ell_k+1}$. Hence the Paley k-hypergraph $P_{q,k,r}$ exists for each $k \in K$, and so $P_{q,K,r}$ is a well-defined K-hypergraph. Moreover, the subhypergraph X_k of $X = P_{q,K,r}$ induced by the edges of size k is the (vertex transitive) self-complementary k-hypergraph $X_k = P_{q,k,r}$. Hence condition (1) of Proposition 6.1.1 holds. Now if ω is a generator of $\mathbb{F}_q$, then Lemma 4.2.6(3) implies that $\alpha_{\omega^r,0} \in \mathrm{Ant}(P_{q,k,r}) = \mathrm{Ant}(X_k)$ for all $k \in K$. Hence $\bigcap_{k\in K} \mathrm{Ant}(X_k) \neq \emptyset$, and so condition (2) of Proposition 6.1.1 also holds. Thus $P_{q,K,r}$ is a self-complementary K-hypergraph.

Lemma 4.2.6(2) implies that the group $G = \{\alpha_{1,b} : b \in \mathbb{F}_n\} \leq \mathrm{Aut}(P_{q,k,r})$ for all $k \in K$. Hence $\mathrm{Aut}(P_{q,K,r}) = \bigcap_{k\in K} \mathrm{Aut}(P_{q,k,r})$ contains the subgroup G, which acts transitively on $\mathbb{F}_q = V(P_{q,K,r})$. Thus $P_{q,K,r}$ is vertex transitive. ∎

Construction 9.2.1 and Lemma 9.2.2 together imply the following result, which gives some sufficient conditions on the order of a vertex transitive self-complementary K-hypergraph.

Theorem 9.2.3 *Let K be a set of positive integers such that* $\min\{k : k \in K\} \geq 2$. *For each $k \in K$, let $\ell_k = \max\{k_{(2)}, (k-1)_{(2)}\}$. There exists a vertex transitive self-complementary K-hypergraph of order n for every prime power n congruent to 1 modulo $2^{\ell+1}$, where $\ell = \max\{\ell_k : k \in K\}$.*

Note that Theorem 9.2.3 implies that the converse to Theorem 9.1.1 holds in the cases where $t = 1$ and n is a prime power. The next construction for vertex transitive self-complementary K-hypergraphs is an extension of Construction 4.2.8. It shows that the converse of Theorem 9.1.1 holds for all n in the case where $t = 1$ and $L = K$.

Construction 9.2.4 ***Generalized Paley K-hypergraph***

Let K be a set of positive integers such that $\min\{k : k \in K\} \geq 2$. For each $k \in K$, let

$$\ell_k = \max\{k_{(2)}, (k-1)_{(2)}\},$$

and let

$$\ell = \max\{\ell_k : k \in K\}.$$

Let n be a positive integer such that

$$p^{n_{(p)}} \equiv 1 \pmod{2^{\ell+1}} \quad \text{for every prime } p.$$

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ be the unique prime factorization of n, where p_i is prime, $\alpha_i \geq 1$ and $p_i \neq p_j$ for all $i, j \in \{1, 2, \ldots, t\}$ such that $i \neq j$. For each $i \in \{1, 2, \ldots, t\}$, let $q_i = p_i^{\alpha_i}$, let r_i be a divisor of the integer $(q_i - 1)/2^{\ell+1}$, and let $r = (r_1, r_2, \ldots, r_t)$. Let $\mathbb{F}_{q_i}$ denote the field of order q_i. Define $X_{n,K,r}$ to be the K-hypergraph with vertex set

$$V(X_{n,K,r}) = \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \times \cdots \times \mathbb{F}_{q_t}$$

and edge set

$$E(X_{n,K,r}) = \bigcup_{k \in K} E(X_{n,k,r}),$$

where for each $k \in K$, the symbol $X_{n,k,r}$ denotes the k-uniform hypergraph of Construction 4.2.8.

For $i \in \{1, 2, \ldots, t\}$, an element $a \in \mathbb{F}_{q_i}^*$, and an element $b \in \mathbb{F}_{q_i}$, the symbol $\alpha_{i,a,b}$ denotes the permutation $\alpha_{a,b} \in Sym(\mathbb{F}_{q_i})$ defined on page 108.

Lemma 9.2.5 *The K-hypergraph $X_{n,K,r}$ defined in Construction 9.2.4 is a vertex transitive and self-complementary K-hypergraph.*

Proof: Let $i \in \{1, 2, \ldots, t\}$ and let $k \in K$. Since $\ell \geq \ell_k$ and $q_i \equiv 1 \pmod{2^{\ell+1}}$, it follows that $q_i \equiv 1 \pmod{2^{\ell_k+1}}$. Also, since

$$\frac{q_i - 1}{2^{\ell_k+1}} = \frac{q_i - 1}{2^{\ell_k+1}} \frac{2^{\ell-\ell_k}}{2^{\ell-\ell_k}} = \frac{q_i - 1}{2^{\ell+1}} 2^{\ell-\ell_k}$$

and r_i divides $(q_i - 1)/2^{\ell+1}$, it follows that r_i also divides $(q_i - 1)/2^{\ell_k+1}$. Hence the k-hypergraph $X_{n,k,r}$ exists for each $k \in K$, and so $X_{n,K,r}$ is a well-defined K-hypergraph. Moreover, the subhypergraph $X_{n,k,r}$ of $X_{n,K,r}$ induced by the edges of size k is a (vertex transitive) self-complementary k-hypergraph. Hence condition (1) of Proposition 6.1.1 holds.

For each $i \in \{1, 2, \ldots, t\}$, let ω_i be a generator of $\mathbb{F}_{q_i}$. Then Lemma 4.2.6(2) and (3) imply that $\alpha_{i,\omega_i^{r_i},0} \in \mathrm{Ant}(P_{q_i,k,r_i})$ for all $k \in K$, and hence

$$\alpha_{1,\omega_1^{r_1},0} \times \alpha_{2,\omega_2^{r_2},0} \times \cdots \times \alpha_{t,\omega_t^{r_t},0} \in \mathrm{Ant}(X_{n,k,r})$$

for all $k \in K$. Hence $\bigcap_{k \in K} \mathrm{Ant}(X_{n,k,r}) \neq \emptyset$, and so condition (2) of Proposition 6.1.1 also holds. Thus $X_{n,K,r}$ is a self-complementary K-hypergraph.

By Lemma 4.2.6(2), the group $G_i = \{\alpha_{i,1,b} : b \in \mathbb{F}_{q_i}\} \leq \mathrm{Aut}(P_{q_i,k,r_i})$ for all $k \in K$. Hence $G = G_1 \times G_2 \times \cdots \times G_t \leq \mathrm{Aut}(X_{n,k,r})$ for all $k \in K$. Thus $\mathrm{Aut}(X_{n,K,r}) = \bigcap_{k \in K} \mathrm{Aut}(X_{n,k,r})$ contains the subgroup G, which acts transitively on

$$V(X_{n,K,r}) = \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \times \cdots \times \mathbb{F}_{q_t}.$$

Thus $X_{n,K,r}$ is vertex transitive. ∎

Construction 9.2.4 and Lemma 9.2.5 together imply the following result, which gives some sufficient conditions on the order of a vertex transitive self-complementary K-hypergraph.

Theorem 9.2.6 *Let K be a set of positive integers such that* $\min\{k : k \in K\} \geq 2$. *For each $k \in K$, let $\ell_k = \max\{k_{(2)}, (k-1)_{(2)}\}$, and let $\ell = \max\{\ell_k : k \in K\}$. There exists a vertex transitive self-complementary K-hypergraph of order n for every positive integer n such that*

$$p^{n_{(p)}} \equiv 1 \pmod{2^{\ell+1}} \quad \textit{for every prime } p.$$

∎

Note that Theorem 9.2.6 implies that the converse to Theorem 9.1.1 holds in the case where $L = K$.

9.2.2 A rank-increasing construction

We begin by introducing a rank-increasing construction due to Potočnik and Šajna [24] which takes a k-hypergraph as input, and returns a k^*-hypergraph for any $k^* \geq k$. Lemma 9.2.8 shows that if $\binom{k^*}{k}$ is odd, this construction preserves the properties of vertex-transitivity and self-complementarity. The proof of Lemma 9.2.8 is included for the sake of completeness.

Construction 9.2.7 [24]
Let $X = (V, E)$ be a k-hypergraph and let k^* be an integer such that $k^* \geq k$. Define X^* to be the k^*-hypergraph with vertex set $V^* = V$ and edge set

$$E^* = \{e \in V^{(k^*)} : e \text{ contains an even number of elements of } E \text{ as subsets}\}.$$

Lemma 9.2.8 [24] *Let $X = (V, E)$ be a k-hypergraph, let k^* be an integer such that $k^* \geq k$ and $\binom{k^*}{k}$ is odd, and let X^* be the k^*-hypergraph defined in Construction 9.2.7.*

1. *If X is self-complementary, then so is X^*.*

2. *If X is vertex transitive, then so is X^*, and if X is doubly transitive, then so is X^*.*

Proof: Let $e \in V^{(k^*)}$. By definition, we have $e \in E^*$ if and only if e contains an even number of elements of E as subsets. Since $\binom{k^*}{k}$ is odd, this is equivalent to saying that e contains an odd number of elements of $V^{(k)} \setminus E$. Hence $e \notin E^*$ if and only if e contains an even number of elements of $V^{(k)} \setminus E$. This implies that any antimorphism of X is also an antimorphism of X^*, and any automorphism of X is also an automorphism of X^*. Hence if X is self-complementary, then so is X^*. Moreover, the transitivity properties of X are inherited by X^*. Thus if X is vertex-transitive, then so is X^*, and if X is doubly-transitive, then so is X^*. ∎

For a nonempty set K of positive integers, we can use Construction 9.2.7 to construct a K-hypergraph from a uniform hypergraph whose rank is the smallest integer in K.

Construction 9.2.9 Let K be a set of positive integers such that $k_{\min} = \min\{k : k \in K\} \geq 2$, and let $X = (V, E)$ be a $k_{\min}$-hypergraph. For each $k \in K$ such that $k > k_{\min}$, use Construction 9.2.7 to construct the k-hypergraph $X_k^* = (V^*, E_k^*)$ with vertex set $V^* = V$ and edge set

$$E_k^* = \{e \in V^{(k)} : e \text{ contains an even number of elements of } E \text{ as subsets}\}.$$

Now define $X^* = (V^*, E^*)$ to be the K-hypergraph with vertex set $V^* = V$ and edge set $E^* = \bigcup_{k \in K} E_k^*$.

Lemma 9.2.10 *Let K be a set of positive integers such that $k_{\min} = \min\{k : k \in K\} \geq 2$, and let $X = (V, E)$ be a $k_{\min}$-hypergraph. Let X^* be the K-hypergraph defined in Construction 9.2.9. Suppose that $\binom{k}{k_{\min}}$ is odd for all $k \in K$. Then the following hold.*

1. *If X is self-complementary, then so is X^*.*

2. *If X is vertex transitive, then so is X^*, and if X is doubly transitive, then so is X^*.*

Proof: Let $k \in K$. Since $k \geq k_{\min}$ and $\binom{k}{k_{\min}}$ is odd by assumption, the proof of Lemma 9.2.8 shows that $\mathrm{Aut}(X) \leq \mathrm{Aut}(X_k^*)$ and $\mathrm{Ant}(X) \leq \mathrm{Ant}(X_k^*)$. Since k was arbitrary, we conclude that

$$\mathrm{Ant}(X) \leq \bigcap_{k \in K} \mathrm{Ant}(X_k^*) = \mathrm{Ant}(X^*)$$

and

$$\mathrm{Aut}(X) \leq \bigcap_{k \in K} \mathrm{Aut}(X_k^*) = \mathrm{Aut}(X^*).$$

Hence if X is self-complementary, then so is X^*, and the transitivity properties of X are inherited by X^*. Thus if X is vertex transitive, then so is X^*, and if X is doubly transitive, then so is X^*. ∎

Construction 9.2.9 can be used to generate many vertex transitive self-complementary K-hypergraphs for various sets of positive integers K from the vertex transitive self-complementary graphs and 3-hypergraphs which were constructed in Chapter 4. We obtain the following sufficient conditions on the orders of vertex transitive self-complementary K-hypergraphs.

Recall that a *Muzychuk integer* is a positive integer n such that $p^{n_{(p)}}$ is congruent to 1 modulo 4 for all primes p.

Theorem 9.2.11 *Let K be a set of positive integers. Let n be a Muzychuk integer, and let q be a prime power congruent to 1 modulo 4.*

(1) If $k \equiv 2$ or $3 \pmod 4$ for all $k \in K$, then there exists a vertex transitive self-complementary K-hypergraph of order n.

(2) If $k \equiv 3 \pmod 4$ for all $k \in K$, then there exist vertex transitive self-complementary K-hypergraphs of orders $2n$ and $(1+q)n$, and there exists a doubly transitive self-complementary K-hypergraph of order $1+q$.

Proof:

(1) Let n be a Muzychuk integer, and let K be a set of positive integers such that $k \equiv 2$ or $3 (\mathrm{mod}\ 4)$ for all $k \in K$. Let $\hat{K} = K \cup \{2\}$. Then $\hat{k}_{\min} = \min\{k : k \in \hat{K}\} = 2$. By Theorem 4.2.2(1), there exists a vertex transitive self-complementary graph (a 2-hypergraph) $X_2^n = (V, E)$ of order n. Let $X_{\hat{K}}^n$ be the $\hat{K}$-hypergraph of Construction 9.2.9 obtained using the base graph X. Since $k \equiv 2$ or $3 \pmod 4$,

we have $k = 4t + 2$ or $k = 4t + 3$ for some nonnegative integer t, and so

$$\binom{k}{2} = \frac{(4t+2)(4t+1)}{2} = (2t+1)(4t+1),$$

or

$$\binom{k}{2} = \frac{(4t+3)(4t+2)}{2} = (4t+3)(2t+1).$$

Hence in either case $\binom{k}{2}$ is odd. Thus $\binom{k}{2}$ is odd for all $k \in \hat{K}$, and so Lemma 9.2.10 implies that $X_{\hat{K}}^n$ is a vertex transitive self-complementary $\hat{K}$-hypergraph of order n. Since $K \subseteq \hat{K}$, the K-subhypergraph of $X_{\hat{K}}^n$ induced by the edges with ranks in K is a vertex transitive self-complementary K-hypergraph of order n.

(2) Suppose that K is a set of positive integers such that $k \equiv 3 \pmod 4$ for all $k \in K$, let n be a Muzychuk integer, and let q be a prime power congruent to 1 modulo 4. Let $\hat{K} = K \cup \{3\}$.

By Theorem 4.2.3(2), there exists a vertex transitive self-complementary 3-hypergraph X_3^{2n} of order $2n$, a vertex transitive self-complementary 3-hypergraph $X_3^{(1+q)n}$ of order $(1+q)n$, and a doubly transitive self-complementary 3-hypergraph X_3^{1+q} of order $1+q$. Let $X_{\hat{K}}^{2n}$ be the $\hat{K}$-hypergraph of Construction 9.2.9 obtained using base graph X_3^{2n}, let $X_{\hat{K}}^{(1+q)n}$ be the $\hat{K}$-hypergraph of Construction 9.2.9 obtained using base graph $X_3^{(1+q)n}$, and let $X_{\hat{K}}^{1+q}$ be the $\hat{K}$-hypergraph of Construction 9.2.9 obtained using base graph X_3^{1+q}.

Since $k \equiv 3 \pmod 4$, we have $k = 4t+3$ for some nonnegative integer t. Thus

$$\binom{k}{3} = \frac{(4t+3)(4t+2)(4t+1)}{3!} = \frac{(4t+3)(2t+1)(4t+1)}{3},$$

which is odd. It follows that $\binom{k}{3}$ is odd for all $k \in \hat{K}$. Thus Lemma 9.2.10 implies that $X_{\hat{K}}^{2n}$ and $X_{\hat{K}}^{(1+q)n}$ are vertex transitive self-complementary $\hat{K}$-hypergraphs of order $2n$ and $(1+q)n$, respectively, and that $X_{\hat{K}}^{1+q}$ is a doubly transitive self-complementary $\hat{K}$-hypergraph of order $1+q$. Since $K \subseteq \hat{K}$, the K-subhypergraphs of $X_{\hat{K}}^{2n}$ and $X_{\hat{K}}^{(1+q)n}$ induced by the edges with sizes in K are vertex transitive self-complementary K-hypergraphs of order $2n$ and $(1+q)n$, respectively, and the K-subhypergraph of $X_{\hat{K}}^{1+q}$ induced by the edges with sizes in K is a doubly transitive self-complementary K-hypergraph of order $1+q$. ∎

Appendix A

In Lemma A.0.13, we will show directly that the necessary and sufficient condition (2.3.1) of Corollary 2.3.2 on the order n of a self-complementary k-hypergraph is equivalent to Szymański and Wojda's condition that $\binom{n}{k}$ is even. First we will need a preliminary lemma.

Recall the definitions of the symbols $n_{[m]}$ and $\left[\frac{n}{m}\right]$ from page 5 of Section 1.1.

Lemma A.0.12 *Let m, n, d be positive integers, where $m \geq n$. Then*

$$\left[\frac{m}{d}\right] - \left[\frac{n}{d}\right] - \left[\frac{m-n}{d}\right] = \begin{cases} 1 & \text{if } m_{[d]} < n_{[d]} \\ 0 & \text{otherwise} \end{cases}.$$

Proof: By the division algorithm, we have

$$m = \left[\frac{m}{d}\right] d + m_{[d]} \quad \text{and} \quad n = \left[\frac{n}{d}\right] d + n_{[d]},$$

where $0 \leq m_{[d]} < d$ and $0 \leq n_{[d]} < d$. Hence

$$m - n = \left(\left[\frac{m}{d}\right] - \left[\frac{n}{d}\right]\right) d + \left(m_{[d]} - n_{[d]}\right), \tag{A.0.1}$$

and $-d < m_{[d]} - n_{[d]} < d$.

If $m_{[d]} \geq n_{[d]}$, then $0 \leq m_{[d]} - n_{[d]} < d$ and so (A.0.1) shows that $(m-n)_{[d]} = m_{[d]} - n_{[d]}$ and $\left[\frac{m-n}{d}\right] = \left[\frac{m}{d}\right] - \left[\frac{n}{d}\right]$, which implies that

$$\left[\frac{m}{d}\right] - \left[\frac{n}{d}\right] - \left[\frac{m-n}{d}\right] = 0.$$

On the other hand, if $m_{[d]} < n_{[d]}$, then we can rewrite (A.0.1) as

$$m - n = \left(\left[\frac{m}{d} \right] - \left[\frac{n}{d} \right] - 1 \right) d + \left(d + m_{[d]} - n_{[d]} \right),$$

where $0 < d + m_{[d]} - n_{[d]} < d$, which shows that $(m-n)_{[d]} = d + m_{[d]} - n_{[d]}$ and $\left[\frac{m-n}{d}\right] = \left[\frac{m}{d}\right] - \left[\frac{n}{d}\right] - 1$. This implies that

$$\left[\frac{m}{d} \right] - \left[\frac{n}{d} \right] - \left[\frac{m-n}{d} \right] = 1.$$

∎

Recall the definition of the symbol $n_{(p)}$ from page 5 of Section 1.1. It is well known that for any positive integer m and prime number p, we have

$$m!_{(p)} = \sum_{r \geq 1} \left[\frac{m}{p^r} \right].$$

It follows that

$$\begin{aligned} \binom{m}{n}_{(p)} &= \left(\frac{m!}{n!(m-n)!} \right)_{(p)} \\ &= m!_{(p)} - n!_{(p)} - (m-n)!_{(p)} \\ &= \sum_{r \geq 1} \left\{ \left[\frac{m}{p^r} \right] - \left[\frac{n}{p^r} \right] - \left[\frac{m-n}{p^r} \right] \right\}. \end{aligned} \tag{A.0.2}$$

We can evaluate each term in the sum above using the fact that

$$\left[\frac{m}{p^r} \right] - \left[\frac{n}{p^r} \right] - \left[\frac{m-n}{p^r} \right] = \begin{cases} 1 & \text{if } m_{[p^r]} < n_{[p^r]} \\ 0 & \text{otherwise} \end{cases}, \tag{A.0.3}$$

which follows directly from Lemma A.0.12.

Lemma A.0.13

Let k and n be positive integers, $k \leq n$, and let b be the binary representation of k. Then $\binom{n}{k}$ is even if and only if

$$n_{[2^{\ell+1}]} < k_{[2^{\ell+1}]} \textit{ for some } \ell \in supp(b). \tag{A.0.4}$$

Proof: Observe that $\binom{n}{k}$ is even if and only if $\binom{n}{k}_{(2)} \geq 1$. By (A.0.2) we have

$$\binom{n}{k}_{(2)} = \sum_{r \geq 1} \left\{ \left[\frac{n}{2^r}\right] - \left[\frac{k}{2^r}\right] - \left[\frac{n-k}{2^r}\right] \right\}. \tag{A.0.5}$$

By (A.0.3), for each $r \geq 1$ we have

$$\left[\frac{n}{2^r}\right] - \left[\frac{k}{2^r}\right] - \left[\frac{n-k}{2^r}\right] = \begin{cases} 1 & \text{if } n_{[2^r]} < k_{[2^r]} \\ 0 & \text{otherwise} \end{cases}.$$

Hence (A.0.5) implies that $\binom{n}{k}$ is even if and only if

$$n_{[2^r]} < k_{[2^r]} \quad \text{for some } r \geq 1,$$

that is, if and only if

$$n_{[2^{\ell+1}]} < k_{[2^{\ell+1}]} \quad \text{for some } \ell \geq 0. \tag{A.0.6}$$

Now we will show that the condition in (A.0.6) holds for some $\ell \geq 0$ if and only if it holds for some $\ell \in \operatorname{supp}(b)$. If (A.0.6) holds for some $\ell \in \operatorname{supp}(b)$, then (A.0.6) certainly holds for some $\ell \geq 0$. Conversely, assume for the sake of contradiction that the condition in (A.0.6) does not hold for any $\ell \in \operatorname{supp}(b)$, but it holds for some $\ell \notin \operatorname{supp}(b)$. Now if $i \notin \operatorname{supp}(b)$ for all i such that $0 \leq i \leq \ell$, then $k_{[2^{\ell+1}]} = \sum_{i=0}^{\ell} b_i 2^i = 0$, and so (A.0.6) implies that $n_{[2^{\ell+1}]} < 0$, giving a contradiction. Hence there must exist a nonnegative integer $i < \ell$ such that $i \in \operatorname{supp}(b)$. Let ℓ_* denote the largest such integer i. Then $k_{[2^{\ell+1}]} = \sum_{i=0}^{\ell_*} b_i 2^i = k_{[2^{\ell_*+1}]}$, and so (A.0.6) implies that

$$n_{[2^{\ell+1}]} < k_{[2^{\ell_*+1}]}. \tag{A.0.7}$$

Since $\ell_* < \ell$, we have $n_{[2^{\ell_*+1}]} \leq n_{[2^{\ell+1}]}$, and so (A.0.7) implies that

$$n_{[2^{\ell_*+1}]} < k_{[2^{\ell_*+1}]}.$$

Hence $\ell_* \in \operatorname{supp}(b)$ and (A.0.6) holds for ℓ_*, contradicting our assumption. We conclude that (A.0.6) holds if and only if $n_{[2^{\ell+1}]} < k_{[2^{\ell+1}]}$ for some $\ell \in \operatorname{supp}(b)$, and thus

$\binom{n}{k}$ is even if and only if (A.0.4) holds. ∎

The following technical lemma is used in the proof of Theorem 2.2.5 in Section 2.2.2. It is also used to verify the validity of Algorithm 2.4.2 in Section 2.4.

Lemma A.0.14 *Let ℓ and n be positive integers such that $n \geq 2$. If there exists a sequence of nonnegative integers $a_0, a_1, \ldots, a_{\ell-1}$ such that $\sum_{i=0}^{\ell-1} a_i n^i \geq n^\ell$, then there exists a sequence of integers $c_0, c_1, \ldots, c_{\ell-1}$ such that $0 \leq c_i \leq a_i$ for $i = 0, 1, \ldots, \ell-1$, and $\sum_{i=0}^{\ell-1} c_i n^i = n^\ell$.*

Proof: The proof is by induction on ℓ.

Base Step: The statement is certainly true if $\ell = 1$, for if there is a nonnegative integer a_0 such that $a_0 n^0 \geq n^1 = n$, then $a_0 \geq n$, and so the result holds with $c_0 = n$.

Induction Step: Let $\ell \geq 2$ and assume that the statement is true for $\ell - 1$. That is, assume that if there is a sequence of non-negative integers $\hat{a}_0, \ldots, \hat{a}_{\ell-2}$ such that $\sum_{i=0}^{\ell-2} \hat{a}_i n^i \geq n^{\ell-1}$, then there exists a sequence of integers $\hat{c}_0, \ldots, \hat{c}_{\ell-2}$ with $0 \leq \hat{c}_i \leq \hat{a}_i$, for $i = 0, 1, \ldots, \ell-2$, such that $\sum_{i=0}^{\ell-2} \hat{c}_i n^i = n^{\ell-1}$.

Now suppose that $a_0, \ldots, a_{\ell-1}$ is a sequence of nonnegative integers such that $\sum_{i=0}^{\ell-1} a_i n^i \geq n^\ell$. If $a_{\ell-1} \geq n$, then to obtain the desired sequence, set $c_i = 0$ for all $i \in \{0, 1, \ldots, \ell-2\}$, and set $c_{\ell-1} = n$. Then $0 \leq c_i \leq a_i$ for all i, and $\sum_{i=0}^{\ell-1} c_i n^i = n^\ell$, as required.

Hence we may assume that $a_{\ell-1} \leq n-1$. Then $a_{\ell-1} = n - k$ for an integer k such that $1 \leq k \leq n$. In this case $a_0, a_1, \ldots, a_{\ell-2}$ is a sequence such that

$$\sum_{i=0}^{\ell-2} a_i n^i \geq n^\ell - (n-k)n^{\ell-1} = kn^{\ell-1} \geq n^{\ell-1}.$$

Hence by the induction hypothesis, there exists a sequence $\{c_i^1\}$ such that $0 \leq$

$\sum_{j=0}^{k} c_i^1 \leq a_i$ for all $i \in \{0, 1, \ldots, \ell - 2\}$, and $\sum_{i=0}^{\ell-2} c_i^1 n^i = n^{\ell-1}$. Now

$$\sum_{i=0}^{\ell-2}(a_i - c_i^1)n^i \geq n^\ell - (n-k+1)n^{\ell-1} = (k-1)n^{\ell-1} \geq n^{\ell-1}$$

for $k > 1$. Thus for $k > 1$ we can continue in this way, applying the induction hypothesis k times to obtain k sequences of integers $\{c_i^1\}, \{c_i^2\}, \ldots, \{c_i^k\}$ such that $0 \leq \sum_{j=0}^{k} c_i^j \leq a_i$ for all $i \in \{0, 1, \ldots, \ell-2\}$, and $\sum_{i=0}^{\ell-2} c_i^j n^i = n^{\ell-1}$ for all $j \in \{1, 2, \ldots, k\}$. Now to obtain the desired sequence, set $c_i = \sum_{j=1}^{k} c_i^j$ for all $i \in \{0, 1, \ldots, \ell - 2\}$, and set $c_{\ell-1} = a_{\ell-1} = n - k$. Then certainly $0 \leq c_i \leq a_i$ for $i = 0, 1, \ldots, \ell - 1$. Moreover,

$$\begin{aligned}
\sum_{i=0}^{\ell-1} c_i n^i &= \sum_{i=0}^{\ell-2} c_i n^i + c_{\ell-1} n^{\ell-1} \\
&= \sum_{i=0}^{\ell-2} \left[\sum_{j=1}^{k} c_i^j\right] n^i + (n-k)n^{\ell-1} \\
&= \sum_{j=1}^{k} \left[\sum_{i=0}^{\ell-2} c_i^j n^i\right] + (n-k)n^{\ell-1} \\
&= \sum_{j=1}^{k} n^{\ell-1} + (n-k)n^{\ell-1} \\
&= kn^{\ell-1} + (n-k)n^{\ell-1} = n^\ell,
\end{aligned}$$

as required. The result follows by induction. ∎

The next two technical lemmas are used in the proof of Lemma 3.2.3 in Section 3.2.2.

Lemma A.0.15 *Let α, i, and j be integers such that $0 \leq i, j \leq 2^\alpha$, $i + j$ is odd, and $\frac{1}{2^\alpha}\binom{2^\alpha}{i}\binom{2^\alpha}{j}$ is an integer. Then $\frac{1}{2^\alpha}\binom{2^\alpha}{i}\binom{2^\alpha}{j}$ is odd if and only if $i \in \{0, 2^\alpha\}$ or $j \in \{0, 2^\alpha\}$.*

Proof: Since $i + j$ is odd, either i or j must be odd. First suppose that i is odd.

Now

$$\frac{1}{2^\alpha}\binom{2^\alpha}{i}\binom{2^\alpha}{j} = \frac{1}{2^\alpha - i}\binom{2^\alpha - 1}{i}\binom{2^\alpha}{j}.$$

Since $i \leq 2^\alpha$ and i is odd, any integer r in the support of the binary representation of i satisfies $r \leq \alpha - 1$. Since $(2^\alpha - 1)_{[2^{r+1}]} \geq i_{[2^{r+1}]}$ for all such r, Lemma A.0.13 implies that $\binom{2^\alpha-1}{i}$ is odd. Now if $j \notin \{0, 2^\alpha\}$, then $(2^\alpha)_{[2^{r+1}]} = 0 < j_{[2^{r+1}]}$ for some r in the support of the binary representation of j, and so Lemma A.0.13 implies that $\binom{2^\alpha}{j}$ is even. Since $2^\alpha - i$ is odd, this implies that the integer $\frac{1}{2^\alpha}\binom{2^\alpha}{i}\binom{2^\alpha}{j} = \frac{1}{2^\alpha-i}\binom{2^\alpha-1}{i}\binom{2^\alpha}{j}$ is even. On the other hand, if $j \in \{0, 2^\alpha\}$, then $\binom{2^\alpha}{j} = 1$, and so $\frac{1}{2^\alpha}\binom{2^\alpha}{i}\binom{2^\alpha}{j} = \frac{1}{2^\alpha-i}\binom{2^\alpha-1}{i}\binom{2^\alpha}{j} = \frac{1}{2^\alpha-i}\binom{2^\alpha-1}{i}$ is an odd integer. Thus if i is odd, then $\frac{1}{2^\alpha}\binom{2^\alpha}{i}\binom{2^\alpha}{j}$ is odd if and only if $j \in \{0, 2^\alpha\}$.

By a symmetric argument, if j is odd, then $\frac{1}{2^\alpha}\binom{2^\alpha}{i}\binom{2^\alpha}{j}$ is odd if and only if $i \in \{0, 2^\alpha\}$. ∎

Lemma A.0.16 *Let r be a nonnegative integer. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_n$ is a sequence of integers such that $0 \leq \lambda_i \leq r$ for all $i \in \{1, 2, \ldots, n\}$. Then there is a function $v : \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \to \{-1, 1\}$ such that $0 \leq \sum_{i=1}^{n} \lambda_i v(\lambda_i) \leq r$.*

Proof: The proof is by induction on n. If $n = 1$, then take $v(\lambda_1) = 1$. If $n = 2$, then if $\lambda_1 \leq \lambda_2$, take $v(\lambda_1) = -1$ and $v(\lambda_2) = 1$, and if $\lambda_1 \geq \lambda_2$, take $v(\lambda_1) = 1$ and $v(\lambda_2) = -1$. Hence the result holds when $n \in \{1, 2\}$.

Let $n > 2$ and suppose the result holds for all such sequences of length $n-1$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be a sequence of integers such that $0 \leq \lambda_i \leq r$ for all $i \in \{1, 2, \ldots, n\}$. By the induction hypothesis, there is a function $v' : \{\lambda_1, \ldots, \lambda_{n-1}\} \to \{-1, 1\}$ such that $\sum_{i=1}^{n-1} \lambda_i v'(\lambda_i) = \lambda$ for some λ such that $0 \leq \lambda \leq r$. By the base case $n = 2$, there is a function $\hat{v} : \{\lambda, \lambda_n\} \to \{-1, 1\}$ such that $0 \leq \lambda\hat{v}(\lambda) + \lambda_n\hat{v}(\lambda_n) \leq r$. Let v be the function $v : \{\lambda_1, \ldots, \lambda_n\} \to \{-1, 1\}$ such that $v(\lambda_i) = \hat{v}(\lambda)v'(\lambda_i)$ for $i \in \{1, 2, \ldots, n-1\}$, and $v(\lambda_n) = \hat{v}(\lambda_n)$. Then $\sum_{i=1}^{n} \lambda_i v(\lambda_i) = \lambda\hat{v}(\lambda) + \lambda_n\hat{v}(\lambda_n)$ and so $0 \leq \sum_{i=1}^{n} \lambda_i v(\lambda_i) \leq r$ as required. The result follows by induction. ∎

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