Cyclic partitions of complete and almost complete uniform hypergraphs

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Abstract

We consider cyclic partitions of the complete k-uniform hypergraph on a finite set V, minus a set of s edges, $s \geq 0$. An *s-almost t-complementary k-hypergraph* is a k-uniform hypergraph with vertex set V and edge set E for which there exists a permutation $\theta \in Sym(V)$ such that the sets $E, E^{\theta}, E^{\theta^2}, \ldots, E^{\theta^{t-1}}$ partition the set of all k-subsets of V minus a set of s edges. Such a permutation θ is called an *s-almost (t,k)-complementing permutation*. The s-almost t-complementary k-hypergraphs are a natural generalization of the almost self-complementary graphs which were previously studied by Clapham, Kamble et al, and Wojda. We prove the existence of an s-almost p^{α}-complementary k-hypergraph of order n, where p is prime, $s = \prod_{i\geq 0}\binom{n_i}{k_i}$, and n_i and k_i are the entries in the base p^{α}-representations of n and k, respectively. This existence result yields a combinatorial argument which generalizes Lucas' classic 1878 number theory result to prime powers, which was originally proved by Davis and Webb in 1990 by another method. In addition, we prove an alternative statement of the necessary and sufficient conditions for the existence of a p^{α}-complementary k-hypergraph, and the equivalence of these two conditions yield an interesting relationship between the base-p representation and the base p^{α}-representation of a positive integer n. Finally, we determine a set of necessary and sufficient conditions on n for the existence of a t-complementary k-uniform hypergraph on n vertices for composite values of t, extending previous results due to Wojda, Szymański and Gosselin.

1 Definitions

For a finite set V and a positive integer k, let $\binom{V}{k}$ denote the set of all k-subsets of V. A *hypergraph* with vertex set V and edge set E is a pair (V, E), in which V is a finite set and E is a collection of subsets of V. A hypergraph (V, E) is called *k-uniform* (or a *k-hypergraph*) if E is a subset of $\binom{V}{k}$. Note that a 2-hypergraph is a *graph*. The parameters k and $|V|$ are called the *rank* and the *order* of the k-hypergraph, respectively. The vertex set and the edge set of a hypergraph X will often be denoted by $V(X)$ and $E(X)$, respectively. For a permutation θ of V, we define the induced mapping θ^* on $\binom{V}{k}$ by the formula $\theta^*(e) = \{\theta(v) : v \in e\}$, for every $e \in \binom{V}{k}$. For $E \subset \binom{V}{k}$, we define $E^\theta = \{\theta^*(e) : e \in E\}$. An *isomorphism* between k-hypergraphs X and X' is a bijection $\theta : V(X) \to V(X')$ which induces a bijection θ^* from $E(X)$ to $E(X')$. If such an isomorphism exists, the hypergraphs X and X' are said to be *isomorphic*.

A k-hypergraph $X = (V, E)$ is *s-almost t-complementary* if there exists a permutation θ on V such that the sets $E, E^\theta, E^{\theta^2}, \ldots, E^{\theta^{t-1}}$ partition $\binom{V}{k} - S$, where $S \subset \binom{V}{k} - E$ with $|S| = s$. Such a permutation θ is called an *s-almost (t,k)-complementing permutation*, and it gives rise to a family of t isomorphic k-hypergraphs $\{X_i = (V, E^{\theta^i}) : i = 0, 1, \ldots, t-1\}$ which partition the complete k-uniform hypergraph on V minus a set of s edges, and which are permuted cyclically under the action of θ. When $s = 0$, we obtain the *(cyclically) t-complementary k-uniform hypergraphs* which were studied in [8, 16]. A 0-almost (t,k)-complementing permutation is also called a *(t,k)-complementing permutation*. When $s = 1$, we obtain the *almost t-complementary uniform hypergraphs* studied in [20, 7], and a 1-almost (t,k)-complementing permutation is also called an *almost (t,k)-complementing permutation*. The s-almost t-complementary k-uniform hypergraphs are a natural generalization of self-complementary graphs, which are special cases of s-almost t-complementary k-hypergraphs where $s = 0$ and $t = k = 2$.

For positive integers t, n and m, the symbol $C_t(n)$ denotes the largest integer α such that t^α divides n, and the symbol $n_{[m]}$ denotes the remainder upon division of n by m. If A is a finite set, we write $C_t(A)$ instead of $C_t(|A|)$.

2 History and statement of the main result

In 1978, M.J. Colbourn and C.J. Colbourn [4] showed that one of the most important problems in graph theory, the graph isomorphism problem, is polynomially equivalent to the problem of determining whether two self-complementary graphs are isomorphic. This result inspired a great deal of research into self-complementary graphs and their generalizations, and Farrugia [6] has written a comprehensive reference manual on this topic. Rather than looking at the problem of determining whether a given k-hypergraph is t-complementary, or s-almost t-complementary, we instead consider the problem of which permutations on n vertices are (t,k)-complementing, or s-almost (t,k)-complementing.

There has been much previous work on this problem. Whether or not a permutation θ is (t,k)-complementing depends entirely on the cycle type of θ. The cycle type of $(2,2)$-complementing permutations was characterized in [13, 14] and the cycle types of the $(2,3)$- and $(2,4)$-complementing permutations were characterized in [15] and [18].These results were generalized to characterize the cycle type of the $(2,k)$-complementing permutations in [9, 17, 19, 21], and the cycle type of the $(t,2)$-complementing permutations was determined in [1, 2]. The almost self-complementary graphs were studied by Clapham et al in [3], and the almost self-complementary 3-uniform hypergraphs were studied in [10]. More recently, Wojda generalized their results in [20] and showed that an almost self-complementary k-uniform hypergraph exists if and only if $\binom{n}{k}$ is odd.

When $s=0$, the s-almost t-complementary k-hypergraphs are the t-complementary k-hypergraphs which were studied in 2010 in [8, 16]. They characterized the cycle type of the (t,k)-complementing permutations in $\mathrm{Sym}(n)$ in the case where t is a prime power.

Proposition 2.1. **[8, 16]** *Let n,k,p and α be positive integers such that $k<n$ and p is prime. Let θ be a permutation in $Sym(n)$ with orbits $\mathcal{O}_1,\mathcal{O}_2,\ldots,\mathcal{O}_m$. Then θ is a (p^α,k)-complementing permutation if and only if there is an integer $\ell\geq 0$ such that the union of the orbits of θ with cardinality not divisible by $p^{\ell+\alpha}$ has cardinality less than $k \bmod p^{\ell+1}$. That is,*

$$\sum_{i:C_p(\mathcal{O}_i)<\ell+\alpha}|\mathcal{O}_i|<k_{[p^{\ell+1}]}.$$

Proposition 2.1 yields the following necessary and sufficient conditions on the order of a p^α-complementary k-hypergraph for a prime p.

Proposition 2.2. **[8, 16]** *Let k, n and α be positive integers, $k\leq n$, let p be a prime. There exists a p^α-complementary k-hypergraph of order n if and only if there exists a nonnegative integer ℓ such that*

$$n_{[p^{\ell+\alpha}]}<k_{[p^{\ell+1}]}.$$

In Section 4, we will generalize this result and find necessary and sufficient conditions on the order of a t-complementary k-hypergraph in the case where t is composite (see Theorem 4.2).

If there exists an s-almost t-complementary k-hypergraph $X=(V,E)$ of order n, then there exists a permutation θ on V such that the sets $E,E^\theta,E^{\theta^2},\ldots,E^{\theta^{t-1}}$ partition $\binom{V}{k}-S$, where $S\subset\binom{V}{k}-E$ with $|S|=s$, and so it follows that $\binom{n}{k}\equiv\ s \bmod (t)$.

In the case where $t=p^\alpha$ for a prime p and a positive integer α and $\gcd(p^\alpha,s)=1$, this implies that $C_{p^\alpha}\left(\binom{n}{k}\right)=0$. The following classic result by Kummer from 1852 yields conditions on the base p representations of n and k for which $C_p\left(\binom{n}{k}\right)=0$.

Theorem 2.3. (Kummer) [11](pages 115-116). *Let p be a prime, and let k and n be nonnegative integers such that $k\leq n$. The exponent of the highest* *power of p dividing $\binom{n}{k}$ is the number of borrows involved in subtracting k from n in base-p.*

Another classic result involving binomial coefficients and primes is Lucas' Theorem of 1878.

Theorem 2.4. (Lucas) [12] *Let p be a prime, and let $0 \leq k \leq n$. Then*

$$\binom{n}{k} \equiv \prod_{i \geq 0} \binom{n_i}{k_i} \mod p.$$

Lucas' classic result was later generalized to prime powers in 1990 by Davis and Webb [5]. In Section 3.1, we will present an alternative combinatorial proof of this generalized version of Lucas' Theorem.

If $n = \sum_{i \geq 0} n_i p^i$ and $k = \sum_{i \geq 0} k_i p^i$ are the base-p representations of n and k, respectively, then Kummer's Theorem implies that $C_p\left(\binom{n}{k}\right) = 0$ if and only if $k_i \leq n_i$ for all $i \geq 0$. In the special case where $k_i = n_i$ whenever $k_i \neq 0$, then we say that the base-p representation of k is a *subsequence* of the base-p representation of n. In [7] it was shown that this is a sufficient condition for the existence of an almost p-complementary k-hypergraph of order n, as stated in the following result.

Proposition 2.5. [7] *Let k and n be positive integers, $k < n$, and let p be prime. There exists an almost p-complementary k-hypergraph of order n whenever the base-p representation of k is a subsequence of the base-p representation of n.*

In the next section, we will prove the following more general result, from which both Proposition 3.2 and Proposition 2.2 may be obtained as special cases.

Theorem 2.6. *Let α, k and n be positive integers, $k < n$, let p be prime, and let $n = \sum_{i \geq 0} n_i p^{\alpha i}$ and $k = \sum_{i \geq 0} k_i p^{\alpha i}$ be the base-p^α representations of n and k, respectively. There exists an s-almost p^α-complementary k-hypergraph of order n, where $s = \prod_{i \geq 0} \binom{n_i}{k_i}$.*

To prove Theorem 2.6 we will make use of the following useful lemma proved by Szymański and Wojda in 2010, which follows easily from Lemmas 1 and 2 in [16].

Lemma 2.7. [16] *Let p be a prime, let α, ℓ and m be positive integers, $\ell < m$, and let $\theta = (1, 2, \ldots, m)$ be a cyclic permutation. If $C_p(m) \geq C_p(\ell) + \alpha$ then for every ℓ-subset $e \subset \{1, 2, \ldots, m\}$ we have $(\theta^*)^j(e) \neq e$ whenever $j \not\equiv 0 \pmod{p^\alpha}$.*

After the proof of Theorem 2.6, in Section 3.1 we will look at some important consequences, including an alternate proof of the generalization of Lucas' Theorem to prime powers (see Corollaries 3.6).

3 Proof of Theorem 2.6

Let

$$n = \sum_{i \geq 0} n_i p^{\alpha i} \quad \text{and} \quad k = \sum_{i \geq 0} k_i p^{\alpha i}$$

be the base-p^α representations of n and k, respectively. We construct an s-almost p^α-complementary k-hypergraph of order n, where $s = \prod_{i \geq 0} \binom{n_i}{k_i}$.

Let V be a set of cardinality n and let $V = \bigcup_{i \geq 0} V_i$ be a partition of V such that $|V_i| = n_i p^{\alpha i}$. Furthermore, for each integer $i \geq 0$, let $\bigcup_{j=1}^{n_i} V_{ij}$ be a partition of V_i, where $|V_{ij}| = p^{\alpha i}$ for each $j = 1, 2, \ldots, n_i$. Say $V_{ij} = \{v_1^j, v_2^j, \ldots, v_{p^{\alpha i}}^j\}$. Now each $i \geq 0$ and each $j = 1, 2, \ldots, n_i$, set

$$\theta_{ij} = (v_1^j, v_2^j, \ldots, v_{p^{\alpha i}}^j),$$

set

$$\theta_i = \prod_{j=1}^{n_i} \theta_{ij}$$

and set

$$\theta = \prod_{i \geq 0} \theta_i.$$

Then $\theta \in Sym(V)$ and θ has exactly n_i cycles of length $p^{\alpha i}$, for each nonnegative integer i in the support of the base p^α-representation of n. We show that θ is an *s-almost (p, k)-complementing permutation*.

Let S be the set of k-element subsets of V which are invariant under θ. We claim that $|S| = s$. To see this, observe that each k-subset $e \in S$ is fixed set-wise by θ, and hence each $e \in S$ is a union of orbits of θ. By the uniqueness of the base-p^α representation of k, a k-subset of V is a union of orbits of θ if and only if it is a union of the elements from exactly k_i cycles of length $p^{\alpha i}$ in the disjoint cycle representation of θ, for each nonnegative integer i in the support of the base-p^α-representation of k. Since there are n_i cycles of length $p^{\alpha i}$ in θ, the number of different k-element subsets which are equal to unions of orbits of θ is $\prod_{i \geq 0} \binom{n_i}{k_i}$. Hence $|S| = s$, as claimed.

Since each $e \in S$ consists of a union of orbits of θ, we have $\theta^*(e) = e$. However, for any $e \in \binom{V}{k} - S$, the uniqueness of the base-p^α representation of k guarantees that there is an integer $i_0 \geq 0$ and an integer $j_0 \in \{1, 2, \ldots, n_{i_0}\}$ such that $e \cap V_{i_0 j_0} \neq \emptyset$ and $e \neq V_{i_0 j_0}$. It follows that

$$C_p\left(e \cap V_{i_0 j_0}\right) + \alpha \leq C_p\left(V_{i_0 j_0}\right).$$

Now Lemma 2.7 implies that $(\theta^*)^j(e \cap V_{i_0 j_0}) \neq e \cap V_{i_0 j_0}$, and hence $(\theta^*)^j(e) \neq e$, whenever $j \not\equiv 0 \pmod{p^\alpha}$.

Now we will describe an algorithm which uses the permutation θ to construct some s-almost p^α-complementary k-hypergraphs with vertex set V.

Algorithm 3.1. (I) Construct the orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_m$ of θ on $\binom{V}{k} - S$. Each orbit $\mathcal{O}_j$ has the form

$$e, (\theta^*)(e), (\theta^*)^2(e), (\theta^*)^3(e), \ldots$$

where $e \in \binom{V}{k} - S$.

(II) For each $\ell \in \{1, 2, \ldots, m\}$ and each $r = 0, 1, \ldots, p^\alpha - 1$, let E_r^ℓ denote the set of k-sets of the form $(\theta^*)^{p^{\alpha z}+r}(e)$ in the orbit $\mathcal{O}_\ell$ constructed in (I), where z is an integer. Since $(\theta^*)^j(e) \neq e$ whenever $j \not\equiv 0 \pmod{p^\alpha}$, each orbit $\mathcal{O}_j$ has length divisible by p^α. Thus, within each orbit $\mathcal{O}_\ell$, θ maps E_r^ℓ to E_{r+1}^ℓ for each $i = 0, 1, \ldots, p^\alpha - 2$, and θ maps $E_{p^\alpha-1}^\ell$ to E_0^ℓ.

(III) Let E be a subset of $\binom{V}{k} \setminus S$ that contains exactly one of the sets E_0^ℓ, E_1^ℓ, $E_2^\ell, \ldots, E_{p^\alpha-1}^\ell$ constructed in (II) for each $\ell \in \{1, 2, \ldots, m\}$. Then $X = (V, E)$ is an s-almost p^α-complementary k-hypergraph. Moreover, if there are m orbits of θ on $\binom{V}{k} \setminus S$, then there are $p^{\alpha m}$ different choices for the edge set E, and the $p^{\alpha m}$ different choices for E generate a set of $p^{\alpha m}$ s-almost p^α-complementary k-hypergraphs on V.

∎

3.1 Consequences of Theorem 2.6

In the case where $\alpha = 1$ and the base-p representation of k is a subsequence of the base-p representation of n, we have $n_i = k_i$ whenever $k_i \neq 0$, and so $s = \prod_{i\geq 0} \binom{n_i}{k_i} = 1$, and we obtain Proposition 3.2 as a special case of Theorem 2.6. In the case where $p = 2$, Kummer's Theorem guarantees that $\binom{n}{k}$ is odd (i.e., $C_2(\binom{n}{k}) = 0$) if and only if the base-2 representation of k is a subsequence of the base-2 representation of n, and we obtain Wojda's result in [20] that an almost self-complementary k-hypergraph exists if and only if $\binom{n}{k}$ is odd. Note that $s = 1$ if and only if the base p^α representation of k is a subsequence of the base-p^α representation of n, so Theorem 2.6 guarantees the existence of an almost p^α-complementary k-hypergraph in that case. This yields the following generalization of Proposition 3.2 to prime powers.

Corollary 3.2. *Let k, n and α be positive integers, $k < n$, and let p be prime. There exists an almost p^α-complementary k-hypergraph of order n whenever the base-p^α representation of k is a subsequence of the base-p^α representation of n.*

Whenever $n_i < k_i$ for some $i \geq 0$ in the base-p^α representations of n and k, we have $\binom{n_i}{k_i} = 0$, and so $s = 0$. In that case Theorem 2.6 guarantees the existence of a p^α-complementary k-hypergraph of order n. The next theorem shows that the condition that $n_i < k_i$ for some $i \geq 0$ in the base-p^α representations of

n and k is equivalent to the necessary and sufficient condition for the existence of a p^α-complementary k-hypergraph of order n given in Proposition 2.2.

Theorem 3.3. *Let k, n and α be positive integers, $k \le n$, let p be a prime. Let $n = \sum_{i\ge 0} n_i p^{\alpha i}$ and $k = \sum_{i \ge 0} k_i p^{\alpha i}$ be the base p^α-representations of n and k, respectively. Then there exists an integer $\ell \ge 0$ such that $n_{[p^{\ell+\alpha}]} < k_{[p^{\ell+1}]}$ if and only if $n_i < k_i$ for some $i \ge 0$.*

Proof: If $n_i < k_i$ for some $i \ge 0$, then Theorem 2.6 guarantees the existence of a p^α-complementary k-hypergraph of order n since in that case $s = \prod_{i \ge 0} \binom{n_i}{k_i} = 0$. Hence Proposition 2.2 implies that there exists an integer $\ell \ge 0$ such that $n_{[p^{\ell+\alpha}]} < k_{[p^{\ell+1}]}$.

Conversely, suppose that $n_i \ge k_i$ for all $i \ge 0$. Then

$$n_{[p^{\alpha i}]} = \sum_{j=0}^{i-1} n_j p^{\alpha j} \ge \sum_{j=0}^{i-1} k_j p^{\alpha j} = k_{[p^{\alpha i}]}$$

and so

$$n_{[p^{\alpha i}]} \ge k_{[p^{\alpha i}]} \tag{1}$$

for all $i \ge 0$. Also, observe that for any positive integer m and nonnegative integers x and y, we have

$$m_{[p^y]} \ge m_{[p^x]} \tag{2}$$

whenever $x \le y$. For any nonnegative integer ℓ, the Division Algorithm guarantees unique integers q and r such that $\ell = q\alpha + r$, where $0 \le r < \alpha$. Thus

$$\begin{aligned}
n_{[p^{\alpha+\ell}]} &= n_{[p^{\alpha(q+1)+r}]} & \\
&\ge n_{[p^{\alpha(q+1)}]} & \text{(Eqn. 2: } x = \alpha(q+1),\ y = \alpha(q+1)+r) \\
&\ge k_{[p^{\alpha(q+1)}]} & \text{(Eqn. 1: } i = q+1) \\
&\ge k_{[p^{\alpha q+r+1}]} & \text{(Eqn. 2: } x = \alpha(q+1), y = \alpha q + r + 1, \text{ as } r+1 \le \alpha) \\
&= k_{[p^{\ell+1}]} &
\end{aligned}$$

Hence $n_{[p^{\ell+\alpha}]} \ge k_{[p^{\ell+1}]}$ for every integer $\ell \ge 0$. ∎

Theorem 3.3 shows an interesting relationship between the base-p and the base p^α representations of a positive integer n. Moreover, Theorem 3.3 and Theorem 2.6 together yield the following alternative statement of Proposition 2.2.

Corollary 3.4. *Let k, n and α be positive integers, $k \le n$, let p be a prime, and let $n = \sum_{i \ge 0} n_i p^{\alpha i}$ and $k = \sum_{i\ge 0} k_i p^{\alpha i}$ be the base p^α-representations of n and k, respectively. There exists a p^α-complementary k-hypergraph of order n if and only if $n_i < k_i$ for some $i \ge 0$.*

When $\alpha = 1$, Corollary 3.4 and Kummer's Theorem together imply that the obvious necessary condition on the order n of a p-complementary k-hypergraph, namely that $p | \binom{n}{k}$, is also sufficient.

Corollary 3.5. *Let p be prime, and let n and k be positive integers, $k \leq n$. There exists a p-complementary k-uniform hypergraph of order n if and only if p divides $\binom{n}{k}$.*

Proof: Let $n = \sum_{i\geq 0} n_i p^i$ and $k = \sum_{i\geq 0} k_i p^i$ be the base p-representations of n and k, respectively. By Kummer's Theorem, p divides $\binom{n}{k}$ if and only if there is at least one borrow when subtracting k from n in base p, so $n_i < k_i$ for some integer $i \geq 0$, which holds if and only if there exists a p-complementary k-hypergraph of order n, by Corollary 3.4. ∎

Theorem 2.6 also provides a combinatorial proof of the following generalization of Lucas' Theorem to prime powers.

Corollary 3.6. (Lucas' Theorem for Prime Powers) *Let p be a prime, let α be a positive integer, and let $0 \leq k \leq n$. Let $n = \sum_{i\geq 0} n_i p^{\alpha i}$ and $k = \sum_{i\geq 0} k_i p^{\alpha i}$ be the base p^α-representations of n and k, respectively. Then*

$$\binom{n}{k} \equiv \prod_{i\geq 0} \binom{n_i}{k_i} \mod p^\alpha.$$

Proof: If $k = 0$, the result is clearly true. For $k \geq 1$, Theorem 2.6 guarantees the existence of an s-almost p^α-complementary k-hypergraph of order n for $s = \prod_{i\geq 0} \binom{n_i}{k_i}$, which implies that $\binom{n}{k} \equiv s \mod p^\alpha$, as claimed. ∎

If there exists an s-almost p^α-complementary k-hypergraph on n vertices, then $\binom{n}{k} \equiv s \pmod{p^\alpha}$, and so by Lucas' Theorem for Prime Powers, s must be congruent to $\prod_{i\geq 0} \binom{n_i}{k_i}$, where $n = \sum_{i\geq 0} n_i p^{\alpha i}$ and $k = \sum_{i\geq 0} k_i p^{\alpha i}$ are the base p^α-representations of n and k, respectively. However the value of s need not be equal to $\prod_{i\geq 0} \binom{n_i}{k_i}$.

4 Composite values of t

In this section we will determine necessary and sufficient conditions on the order n of a t-complementary k-uniform hypergraph for composite values of t. We accomplish this by first examining the cycle type of a (t, k)-complementing permutation, for which we have the following natural characterization from [8].

Lemma 4.1. [8] *Let V be a finite set, let k and t be positive integers, and let $\theta \in Sym(V)$. Then the following three statements are equivalent:*

(1) θ is a (t, k)-complementing permutation.

(2) $A^{\theta^j} \neq A$ for $j \not\equiv 0 \pmod t$, for all $A \in \binom{V}{k}$.

(3) The sequence $A, A^\theta, A^{\theta^2}, A^{\theta^3}, \ldots$ has length divisible by t, for all $A \in \binom{V}{k}$.

Suppose that $t = \prod_{j=1}^{r} p_j^{\alpha_j}$ is the prime power decomposition of t, where $p_1, p_2, \ldots, p_r$ are distinct prime powers and $\alpha_1, \alpha_2, \ldots, \alpha_r$ are positive integers. It follows easily from Lemma 4.1 that a permutation $\theta \in Sym(n)$ is (t,k)-complementing if and only if it is also $(p_j^{\alpha_j}, k)$-complementing for each $j = 1, 2, \ldots, r$. Thus Proposition 2.2 implies that, for each $j = 1, 2, \ldots, r$, there is an integer $\ell_j \geq 0$ such that $n_{\left[p_j^{\alpha_j+\ell_j}\right]} < k_{\left[p_j^{\ell_j+1}\right]}$. However, this set of r necessary conditions together are not sufficient for the existence of a (t,k)-complementing permutation, as was pointed out with a counterexample in [16]. The next result shows that if we add an additional set of congruence relations to these r necessary conditions, we obtain a necessary and sufficient set of conditions for the existence of a t-complementary k-uniform hypergraph on n vertices.

Theorem 4.2. *Let n, k and t be positive integers with $k \leq n$ and $t \geq 2$, and let $t = \prod_{j=1}^{r} p_j^{\alpha_j}$ be the prime power decomposition of t, where $p_1, p_2, \ldots, p_r$ are distinct prime powers and $\alpha_1, \alpha_2, \ldots, \alpha_r$ are positive integers. There exists a t-complementary k-hypergraph of order n if and only if the following conditions hold.*

(1) For each $j = 1, 2, \ldots, r$, there is a nonnegative integer $\ell_j \geq 0$ such that $n_{\left[p_j^{\alpha_j+\ell_j}\right]} < k_{\left[p_j^{\ell_j+1}\right]}$.

(2) For each $j = 1, 2, \ldots, r$, there exists a nonnegative integer i_j such that $\alpha_j i_j \geq \alpha_j + \ell_j$ and $n_{\left[t^{i_j}\right]} < k_{\left[p_j^{\ell_j+1}\right]}$.

Proof: Suppose that conditions (1) and (2) hold. Let $n = \sum_{i \geq 0} n_i t^i$ be the base-t representation of n. Let θ be a permutation in $Sym(n)$ which has exactly n_i cycles of length t^i. We will show that θ is $(p_j^{\alpha_j}, k)$-complementing for each j using Proposition 2.1, which implies that θ is also (t,k)-complementing.

Let $j \in \{1, 2, \ldots, r\}$. By condition (1), there is an integer $\ell_j \geq 0$ such that $n_{\left[p_j^{\alpha_j+\ell_j}\right]} < k_{\left[p_j^{\ell_j+1}\right]}$, and by condition (2), there is also an integer $i_j \geq 0$ such that $\alpha_j i_j \geq \alpha_j + \ell_j$ and $n_{\left[t^{i_j}\right]} < k_{\left[p_j^{\ell_j+1}\right]}$. If $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_m$ are the orbits of θ, then

$$\sum_{i : C_{p_j}(\mathcal{O}_i) < \ell_j + \alpha_j} |\mathcal{O}_i| < \sum_{i : C_{p_j}(\mathcal{O}_i) < \alpha_j i_j} |\mathcal{O}_i| = n_{\left[t^{i_j}\right]},$$

where the last equality holds by our construction of θ. Since $n_{\left[t^{i_j}\right]} < k_{\left[p_j^{\ell_j+1}\right]}$, this implies that

$$\sum_{i : C_{p_j}(\mathcal{O}_i) < \ell_j + \alpha_j} |\mathcal{O}_i| < k_{\left[p_j^{\ell_j+1}\right]}.$$

Hence Proposition 2.1 guarantees that θ is $(p_j^{\alpha_j}, k)$-complementing. Since j was arbitrary, we conclude that θ is also (t,k)-complementing in $Sym(n)$. Hence if conditions (1) and (2) hold, there exists a t-complementary k-hypergraph on n vertices.

Now suppose that there exists a t-complementary k-hypergraph on n vertices with (t,k)-complementing permutation θ with orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_m$. Then θ is also $(p_j^{\alpha_j}, k)$-complementing for each $j \in \{1,2,\ldots,r\}$, and so Proposition 2.2 implies that condition (1) holds. It remains to show that condition (2) holds.

Let $j \in \{1,2,\ldots,r\}$. By condition (1), there exists ℓ_j such that $n_{\left[p_j^{\alpha_j+\ell_j}\right]} < k_{\left[p_j^{\ell_j+1}\right]}$. Since θ is a $(p_j^{\alpha_j}, k)$-complementing permutation, Proposition 2.1 implies that

$$\sum_{i:C_{p_j}(\mathcal{O}_i)<\ell_j+\alpha_j} |\mathcal{O}_i| < k_{\left[p_j^{\ell_j+1}\right]}.$$

Let i_j be the smallest nonnegative integer satisfying $\alpha_j i_j \geq \alpha_j + \ell_j$. Now each orbit of θ which has cardinality not divisible by $p^{\alpha_j+\ell_j}$ also has cardinality not divisible by $p_j^{\alpha_j i_j}$, since $\alpha_j i_j \geq \alpha_j + \ell_j$, and each such orbit also has cardinality not divisible by t^{i_j}, since $p_j^{\alpha_j i_j}$ divides t^{i_j}. Hence

$$\sum_{i:C_t(\mathcal{O}_i)<i_j} |\mathcal{O}_i| \leq \sum_{i:C_{p_j}(\mathcal{O}_i)<\ell_j+\alpha_j} |\mathcal{O}_i| < k_{\left[p_j^{\ell_j+1}\right]}.$$

Moreover, since $\sum_{i:C_t(\mathcal{O}_i)<i_j} |\mathcal{O}_i| < k_{\left[p_j^{\ell_j+1}\right]} \leq p_j^{\ell_j+1} \leq t^{i_j}$, we must have

$$n_{\left[t^{i_j}\right]} = \sum_{i:C_t(\mathcal{O}_i)<i_j} |\mathcal{O}_i|.$$

Thus

$$n_{\left[t^{i_j}\right]} < k_{\left[p_j^{\ell_j+1}\right]}.$$

Since j was arbitrary, condition (2) follows. ∎

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