Almost t-complementary uniform hypergraphs

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Abstract

An *almost t-complementary k-hypergraph* is a k-uniform hypergraph with vertex set V and edge set E for which there exists a permutation $\theta \in Sym(V)$ such that the sets $E, E^{\theta}, E^{\theta^2}, \ldots, E^{\theta^{t-1}}$ partition the set of all k-subsets of V minus one edge. Such a permutation θ is called an *almost (t,k)-complementing permutation*. The almost t-complementary k-hypergraphs are a natural generalization of the almost self-complementary graphs which were previously studied by Clapham, Kamble et al, and Wojda. We prove that there exists an almost p-complementary k-hypergraph of order n whenever the base-p representation of k is a subsequence of the base-p representation of n, where p is prime.

1 Definitions

For a finite set V and a positive integer k, let $\binom{V}{k}$ denote the set of all k-subsets of V. A *hypergraph* with vertex set V and edge set E is a pair (V,E), in which V is a finite set and E is a collection of subsets of V. A hypergraph (V,E) is called *k-uniform* (or a *k-hypergraph*) if E is a subset of $\binom{V}{k}$. The parameters k and $|V|$ are called the *rank* and the *order* of the k-hypergraph, respectively. The vertex set and the edge set of a hypergraph X will often be denoted by $V(X)$ and $E(X)$, respectively. Note that a 2-hypergraph is a *graph*. For a permutation θ of V, we define the induced mapping θ^* on $\binom{V}{k}$ by the formula $\theta^*(e) = \{\theta(v) : v \in e\}$, for every $e \in \binom{V}{k}$. For $E \subset \binom{V}{k}$, we define $E^{\theta} = \{\theta^*(e) : e \in E\}$. An *isomorphism* between k-hypergraphs X and X' is a bijection $\theta : V(X) \to V(X')$ which induces a bijection θ^* from $E(X)$ to $E(X')$. If such an isomorphism exists, the hypergraphs X and X' are said to be *isomorphic*.

A k-hypergraph $X = (V,E)$ is *(cyclically) t-complementary* if there exists a permutation θ on V such that the sets $E, E^{\theta}, E^{\theta^2}, \ldots, E^{\theta^{t-1}}$ partition $\binom{V}{k}$.

We denote the set E^{θ^i} by E_i. Note that $E_i^\theta = E_{i+1}$ for $i = 0, 1, \dots, t-2$ and $E_{t-1}^\theta = E_0 = E$. Such a permutation θ is called a *(t,k)-complementing permutation*, and it gives rise to a family of t isomorphic k-hypergraphs $\{X_i = (V, E_i) : i = 0, 1, \dots, t-1\}$ which partition the complete k-hypergraph on V, and which are permuted cyclically under the action of θ. A k-hypergraph $X = (V, E)$ is *almost t-complementary* if there exists a permutation θ on V such that the sets $E, E^\theta, E^{\theta^2}, \dots, E^{\theta^{t-1}}$ partition $\binom{V}{k} - \{e_0\}$, where e_0 is an element of $\binom{V}{k} - E$. Such a permutation θ is called an *almost (t,k)-complementing permutation*.

For a prime p and a positive integer r, the symbol $C_p(r)$ denotes the largest integer α such that p^α divides r.

2 History and statement of the main result

The t-complementatry k-hypergraphs have been previously defined and studied. The 2-complementary 2-hypergraphs are the self-complementary graphs. In 1978, M.J. Colbourn and C.J. Colbourn [4] showed that one of the most important problems in graph theory, the graph isomorphism problem, is polynomially equivalent to the problem of determining whether two self-complementary graphs are isomorphic. Since then, there has been a great deal of research into self-complementary graphs. A good reference on self-complementary graphs and their generalizations was written by A. Farrugia [5]. The 2-complementary k-hypergraphs are the self-complementary k-hypergraphs studied in [7, 12, 14, 15, 17]. The t-complementary graphs (2-hypergraphs) are studied in [1, 2]. Whether or not a permutation θ is (t,k)-complementing depends entirely on the cycle type of θ. In [6], and independently in [13], the cycle type of the (p^α, k)-complementing permutations that have order a power of p are characterized, in the case where p is a prime. This yields necessary and sufficient conditions on the order of a p^α-complementary k-hypergraphs, as well as a method for testing any permutation algorithmically to determine whether it is (p^α, k)-complementing, and a method for generating all of the p^α-complementary k-hypergraphs of order n, for feasible n, up to isomorphism.

The almost 2-complementary k-hypergraphs are the almost self-complementary uniform hypergraphs. Almost self-complementary graphs were studied by Clapham in [3], and almost self-complementary 3-hypergraphs are studied by Kamble et al in [8]. In [16], Wojda obtained the following necessary and sufficient conditions on the order of an almost self-complementary k-uniform hypergraph.

Proposition 2.1. [16] *Let k and n be positive integers, $k < n$. An almost self-complementary k-hypergraph of order n exists if and only if $\binom{n}{k}$ is odd.*

If there exists an almost t-complementary k-hypergraph $X = (V, E)$ of order n, then there exists a permutation θ on V such that the sets $E, E^\theta, E^{\theta^2}, \dots, E^{\theta^{t-1}}$ partition $\binom{V}{k} - \{e_0\}$, and so it follows that $\binom{n}{k} \equiv 1 \bmod (t)$. In the case where $t = p$ for a prime p, this implies that $C_p\left(\binom{n}{k}\right) = 0$. The following classic result by Kummer from 1852 yields conditions on the base p representations of n and k for which $C_p\left(\binom{n}{k}\right) = 0$.

Theorem 2.2. (Kummer) [9](pages 115-116). *Let p be a prime, and let k and n be nonnegative integers such that $k \leq n$. The exponent of the highest power of p dividing $\binom{n}{k}$ is the number of borrows involved in subtracting k from n in base-p.*

If $n = \sum_{i\geq 0} n_i p^i$ and $k = \sum_{i\geq 0} k_i p^i$ are the base-p representations of n and k, respectively, then Kummer's Theorem implies that $C_p\left(\binom{n}{k}\right) = 0$ if and only if $k_i \leq n_i$ for all $i \geq 0$. In the special case where $k_i = n_i$ whenever $k_i \neq 0$, then we say that the base-p representation of k is a *subsequence* of the base-p representation of n. In Section 3, we will show that this is a sufficient condition for the existence of an almost p-complementary k-hypergraph of order n, as stated in the following theorem.

Theorem 2.3. *Let k and n be positive integers, $k < n$, and let p be prime. There exists an almost p-complementary k-hypergraph of order n whenever the base-p representation of k is a subsequence of the base-p representation of n.*

3 Proof of Theorem 2.3

In this section we will prove the main result, Theorem 2.3. First, we will prove the following useful lemma.

Lemma 3.1. *Let p be a prime, let ℓ and m be positive integers, $\ell < m$, and let $\theta = (1, 2, \ldots, m)$ be a cyclic permutation. If $C_p(\ell) < C_p(m)$ then for every ℓ-subset $e \subset \{1, 2, \ldots, m\}$ we have $(\theta^*)^j(e) \neq e$ whenever $j \not\equiv 0 \pmod{p}$.*

Proof: Suppose that $\ell < m$ and that $C_p(\ell) < C_p(m)$. Then $\ell = p^\alpha x$ and $m = p^\beta y$ where $\alpha < \beta$ and $\gcd(x, p) = \gcd(y, p) = 1$. Assume, for the sake of contradiction, that there is an integer $j \not\equiv 0 (\text{mod } p)$ and a subset $\{v_1, v_2, \ldots, v_\ell\} \subset \{1, 2, \ldots, m\}$ such that $(\theta^*)^j(\{v_1, v_2, \ldots, v_\ell\}) = \{v_1, v_2, \ldots, v_\ell\}$. Set $\tau = \theta^j$. Every orbit of τ has the same cardinality, say δ. Then $\ell = \delta\gamma$ where γ is the number of orbits of τ in $\{v_1, v_2, \ldots, v_\ell\}$. This follows because $e = \{v_1, v_2, \ldots, v_\ell\}$ must be equal to a union of orbits of $\tau = \theta^j$ as it is fixed by τ. Let ε denote the identity permutation on $\{1, 2, \ldots, m\}$. We have $\tau^\delta = \varepsilon$ and so

$$\theta^{j\ell} = \tau^{\delta\gamma} = \varepsilon^\gamma = \varepsilon.$$

Since $|\theta| = m$, it follows that $j\ell$ is a multiple of m. Thus $j\ell = mq$ for some integer q, and so $jp^\alpha x = p^\beta yq$. This implies that $jx = p^{\beta-\alpha} yq$. Since p divides neither j nor x, Euclid's Lemma guarantees that $p \nmid jx$, which contradicts our assumption that $\alpha < \beta$. ∎

Proof of Theorem 2.3: Suppose that the base-p representation of k is a subsequence of the base-p representation of n. We construct an almost p-complementary k-hypergraph of order n. Write the base-p representation of k and n including only the nonzero terms in the expansion. Say

$$k = k_1 p^{\alpha_1} + k_2 p^{\alpha_2} + \cdots k_s p^{\alpha_s}$$

$$n = n_1 p^{\beta_1} + n_2 p^{\beta_2} + \cdots n_t p^{\beta_t}$$

where $\alpha_1 < \alpha_2 < \cdots < \alpha_s$ and $\beta_1 < \beta_2 < \cdots < \beta_t$. Since the sequence $\{k_i\}_{i=1}^{s}$ is a subsequence of $\{n_j\}_{j=1}^{t}$, for each $i \in \{1, 2, \ldots, s\}$, there is $j_i \in \{1, 2, \ldots, t\}$ such that $\beta_{j_i} = \alpha_i$ and $n_{j_i} = k_i$.

Let V be a set of cardinality n and let $V = V_1 \cup V_2 \cup \cdots \cup V_t$ be a partition of V such that $|V_j| = n_j p^{\beta_j}$, say $V_j = \{v_i^j, v_2^j, \ldots, v_{n_j p^{\beta_j}}^j\}$, for $j = 1, 2, \ldots, t$. Set $e_0 = V_{j_1} \cup V_{j_2} \cup \cdots \cup V_{j_s}$. Clearly $|e_0| = k$. Now for each $j = 1, 2, \ldots, t$, set

$$\theta_j = (v_i^j, v_2^j, \ldots, v_{n_j p^{\beta_j}}^j)$$

and set

$$\theta = \theta_1 \circ \theta_2 \circ \cdots \circ \theta_t.$$

We show that θ is an *almost* (p, k)*-complementing permutation*.

Since e_0 consists of a union of orbits of θ, we have $\theta^*(e_0) = e_0$. However, for any $e \in \binom{V}{k} - \{e_0\}$, the uniqueness of the base-p representation of k guarantees that there is an index $j_0 \in \{1, 2, \ldots, t\}$ such that $e \cap V_{j_0} \neq \emptyset$ and $e \neq V_{j_0}$. It follows that

$$C_p(|e \cap V_{j_0}|) < C_p(|V_{j_0}|).$$

Now Lemma 3.1 implies that,

$$(\theta^*)^j(e) \neq e \quad \text{whenever} \quad j \not\equiv 0 \pmod{p}.$$

Now we will describe an algorithm which uses the permutation θ to construct some almost p-complementary k-hypergraphs with vertex set V.

Algorithm 3.2. (I) Construct the orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_m$ of θ on $\binom{V}{k} - \{e_0\}$. Each orbit $\mathcal{O}_j$ has the form

$$e, (\theta^*)(e), (\theta^*)^2(e), (\theta^*)^3(e), \ldots$$

where $e \in \binom{V}{k} - \{e_0\}$.

(II) For each $\ell \in \{1, 2, \ldots, m\}$ and each $r = 0, 1, \ldots, p-1$, let E_r^ℓ denote the set of k-sets of the form $(\theta^*)^{pz+r}(e)$ in the orbit $\mathcal{O}_\ell$ constructed in (I), where z is an integer. Since $(\theta^*)^j(e) \neq e$ whenever $j \not\equiv 0 \pmod{p}$, each orbit $\mathcal{O}_j$ has length divisible by p. Thus, within each orbit $\mathcal{O}_\ell$, θ maps E_r^ℓ to E_{r+1}^ℓ for each $i = 0, 1, \ldots, p-2$, and θ maps E_{p-1}^ℓ to E_0^ℓ.

(III) Let E be a subset of $\binom{V}{k}$ that contains exactly one of the sets E_0^ℓ, E_1^ℓ, $E_2^\ell, \ldots, E_{t-1}^\ell$ constructed in (II) for each $\ell \in \{1, 2, \ldots, m\}$. Then $X = (V, E)$ is an almost t-complementary k-hypergraph. Moreover, if there are m orbits of θ on $\binom{V}{k} \setminus \{e_0\}$, then there are p^m different choices for the edge set E, and the p^m different choices for E generate a set of p^m almost t-complementary k-hypergraphs on V.

∎

In [16] Wojda proved this result in the case where $p = 2$ (see Proposition 2.1). His proof used the uniqueness of the base-2 representation of k. He was able to prove the converse as well, in the case where $p = 2$, since in that case Kummer's Theorem guarantees that $\binom{n}{k}$ is odd (i.e., $C_2(\binom{n}{k}) = 0$) if and only if the base-2 representation of k is a subsequence of the base-2 representation of n. For primes $p > 2$, this is not a necessary condition for $C_p(\binom{n}{k}) = 0$, so it is unclear whether the necessary condition that $\binom{n}{k} \equiv 1 \pmod{p}$ is sufficient for the existence of an almost p-complementary k-hypergraph. This remains an open problem, as does the problem of determining necessary and sufficient conditions on the order of almost t-complementary k-hypergraphs for t not prime. Note that Lemma 3.1 does not hold if p is not prime, since then Euclid's Lemma does not apply in the last step, so a different approach is needed to construct almost t-complementary k-hypergraphs for composite values of t. It would also be interesting to characterize the cycle types of almost (t, k)-complementing permutations for t a prime power, to obtain results analagous to those in [6] and [13].

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