Helly Theorems for 3-Steiner and 3-Monophonic Convexity in Graphs

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Abstract

A family \( C \) of sets has the Helly property if any subfamily \( C' \), whose elements are pairwise intersecting, has non-empty intersection. Suppose \( C \) is a non-empty family of subsets of a finite set \( V \). The Helly number \( h(C) \) of \( C \) is the smallest positive integer \( n \) such that every subfamily \( C' \) of \( C \) with \( |C'| \geq n \) and which intersects \( n \)-wise has non-empty intersection.

In this paper we consider the families of convex sets relative to two graph convexities. Suppose \( G \) is a (finite) connected graph and \( U \) a set of vertices of \( G \). Then a connected subgraph with the fewest number of edges containing \( U \) is called a Steiner tree for \( U \), and the collection of all vertices of \( G \) that belong to some Steiner tree for \( U \) is called the Steiner interval for \( U \). A set \( S \) of vertices of \( G \) is \( g_3 \)-convex if it contains the Steiner interval for every 3-subset \( U \) of \( S \). A subtree \( T \) of \( G \) that contains \( U \) is a minimal \( U \)-tree if every vertex of \( T \) that is not in \( U \) is a cut-vertex of the subgraph induced by \( V(T) \). The collection of all vertices that belong to some minimal \( U \)-tree is called the monophonic interval for \( U \) and a set \( S \) of vertices is \( m_3 \)-convex if it contains the monophonic interval of every 3-subset \( U \) of \( S \).

We characterize those (finite) graphs for which the families of convex sets, of cardinality at least 3, with respect to the \( g_3 \)-convexity and \( m_3 \)-convexity have the Helly property. A graph obtained from a complete graph by deleting a matching is called a near-clique. The maximum order of a near-clique in a graph \( G \) is called the near-clique number of \( G \). It is observed that the near-clique number of a graph is a lower bound on the Helly number for both the family of \( g_3 \)- and \( m_3 \)-convex sets. It is shown that the near-clique number of chordal and distance-hereditary graphs equals the Helly number of the \( g_3 \)-convex sets for these graphs and it is shown that there are graphs with near-clique number 3 for which the Helly number of the \( g_3 \)-convex sets is arbitrarily large. For the \( m_3 \)-convex sets it is shown that the near-clique number always equals the Helly number.

Key Words: Helly number, Helly property, Steiner tree, minimal \( U \)-tree, convexity.

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