The repetition threshold for binary rich words

Lucas Mol

THE UNIVERSITY OF WINNIPEG

Joint work with James D. Currie and Narad Rampersad

AMS Special Session on Sequences, Words, and Automata
Joint Mathematics Meetings, Denver, CO
January 15, 2020
Plan

Critical Exponents and Repetition Thresholds

Rich words
FRACTIONAL POWERS

A word \( w = w_1 w_2 \cdots w_n \) has period \( p \) if \( w_i + p = w_i \) for all \( 1 \leq i \leq n - p \).

In this case, the rational number \( n/p \) is called an exponent of \( w \).

If \( w \) has exponent \( r \), then we say that \( w \) is an \( r \)-power.

Example: The word alfalfa is a \( 7/3 \)-power.

Special case: 2-powers are also called squares.
FRACTIONAL POWERS

A word $w = w_1 w_2 \cdots w_n$ has period $p$ if $w_{i+p} = w_i$ for all $1 \leq i \leq n - p$. 

Special case: 2-powers are also called squares.
Fractional Powers

A word $w = w_1 w_2 \cdots w_n$ has period $p$ if $w_{i+p} = w_i$ for all $1 \leq i \leq n - p$.

In this case, the rational number $n/p$ is called an exponent of $w$.
A word $w = w_1 w_2 \cdots w_n$ has *period* $p$ if $w_{i+p} = w_i$ for all $1 \leq i \leq n - p$.

- In this case, the rational number $n/p$ is called an *exponent* of $w$.

- If $w$ has exponent $r$, then we say that $w$ is an *r-power*.
Fractional Powers

- A word $w = w_1 w_2 \cdots w_n$ has period $p$ if $w_{i+p} = w_i$ for all $1 \leq i \leq n - p$.
  - In this case, the rational number $n/p$ is called an exponent of $w$.
- If $w$ has exponent $r$, then we say that $w$ is an $r$-power.
  - Example: The word alfalfa is a $7/3$-power.
FRACTIONAL POWERS

▶ A word $w = w_1 w_2 \cdots w_n$ has period $p$ if $w_{i+p} = w_i$ for all $1 \leq i \leq n - p$.
  
  ▶ In this case, the rational number $n/p$ is called an exponent of $w$.

▶ If $w$ has exponent $r$, then we say that $w$ is an $r$-power.
  
  ▶ Example: The word alfalfa is a 7/3-power.
FRACTIONAL POWERS

▶ A word $w = w_1w_2 \cdots w_n$ has period $p$ if $w_{i+p} = w_i$ for all $1 \leq i \leq n - p$.
  ▶ In this case, the rational number $n/p$ is called an exponent of $w$.

▶ If $w$ has exponent $r$, then we say that $w$ is an $r$-power.
  ▶ Example: The word alfalfa is a 7/3-power.
fractional powers

A word $w = w_1 w_2 \cdots w_n$ has period $p$ if $w_{i+p} = w_i$ for all $1 \leq i \leq n - p$.

- In this case, the rational number $n/p$ is called an exponent of $w$.

If $w$ has exponent $r$, then we say that $w$ is an $r$-power.

- Example: The word alfalfa is a 7/3-power.
FRACTIONAL POWERS

- A word \( w = w_1 w_2 \cdots w_n \) has period \( p \) if \( w_{i+p} = w_i \) for all \( 1 \leq i \leq n - p \).
  - In this case, the rational number \( n/p \) is called an exponent of \( w \).
- If \( w \) has exponent \( r \), then we say that \( w \) is an \( r \)-power.
  - Example: The word alfalfa is a \( 7/3 \)-power.

\[
\begin{pmatrix}
\text{ALFA} \\
\text{LFA} \\
\text{FA} \\
\text{A}
\end{pmatrix}^{7/3} = \left( \begin{array}{c}
\text{ALF} \\
\text{F} \\
\text{L} \\
\text{A}
\end{array} \right)^{7/3}
\]
FRACTIONAL POWERS

► A word $w = w_1 w_2 \cdots w_n$ has period $p$ if $w_{i+p} = w_i$ for all $1 \leq i \leq n - p$.
  ▶ In this case, the rational number $n/p$ is called an exponent of $w$.

► If $w$ has exponent $r$, then we say that $w$ is an $r$-power.
  ▶ Example: The word alfalfa is a 7/3-power.

\[
\begin{align*}
\text{ALFALFA} & = \left( \begin{array}{c}
\text{ALF} \\
\text{ALF} \\
\end{array} \right)^{7/3}
\end{align*}
\]

► Special case: 2-powers are also called squares.
The critical exponent of a word $w$ is defined as $\sup \{ r \in \mathbb{Q} : w \text{ contains an } r\text{-power} \}$.

Let $\mu$ denote the Thue-Morse morphism, defined by $\mu(0) = 01$ and $\mu(1) = 10$.

It is well-known that the Thue-Morse word $\mu(\omega)(0) = 0110100110010110 \cdots$ contains no factors of exponent greater than 2.

It does, however, contain squares.

So the critical exponent of the Thue-Morse word is 2.

The repetition threshold for a set of words $L$ is the smallest critical exponent among all infinite words in $L$.

Since every long enough binary word contains a square, the repetition threshold for the set of all binary words is 2.
The critical exponent of a word $w$ is defined as
\[ \sup \{ r \in \mathbb{Q} : w \text{ contains an } r\text{-power} \}. \]

Let $\mu$ denote the Thue-Morse morphism, defined by $\mu(0) = 01$ and $\mu(1) = 10$.

It is well-known that the Thue-Morse word $\mu^\omega(0) = 0110100110010110 \cdots$ contains no factors of exponent greater than 2.

It does, however, contain squares.

So the critical exponent of the Thue-Morse word is 2.

The repetition threshold for a set of words $L$ is the smallest critical exponent among all infinite words in $L$.

Since every long enough binary word contains a square, the repetition threshold for the set of all binary words is 2.
The critical exponent of a word \( w \) is defined as

\[
\sup \{ r \in \mathbb{Q} : \text{\( w \) contains an \( r \)-power} \}.
\]

Let \( \mu \) denote the Thue-Morse morphism, defined by

\[
\mu(0) = 01 \text{ and } \mu(1) = 10.
\]
The critical exponent of a word \( w \) is defined as

\[
\sup \{ r \in \mathbb{Q} : \text{\( w \) contains an \( r \)-power} \}.
\]

Let \( \mu \) denote the Thue-Morse morphism, defined by

\[
\mu(0) = 01 \quad \text{and} \quad \mu(1) = 10.
\]

It is well-known that the Thue-Morse word

\[
\mu^\omega(0) = 0110100110010110110\ldots
\]

contains no factors of exponent greater than 2.
CRITICAL EXPONENTS AND REPETITION THRESHOLDS

- The critical exponent of a word $w$ is defined as
  \[ \sup \{ r \in \mathbb{Q} : \text{w contains an r-power} \} \].

- Let $\mu$ denote the Thue-Morse morphism, defined by $\mu(0) = 01$ and $\mu(1) = 10$.
  - It is well-known that the Thue-Morse word $\mu^\omega(0) = 0110100110010110 \cdots$ contains no factors of exponent greater than 2.
  - It does, however, contain squares.
The critical exponent of a word $w$ is defined as

$$\sup\{ r \in \mathbb{Q} : \text{w contains an } r\text{-power} \}.$$ 

Let $\mu$ denote the Thue-Morse morphism, defined by $\mu(0) = 01$ and $\mu(1) = 10$.

It is well-known that the Thue-Morse word

$$\mu^\omega(0) = 0110100110010110\ldots$$

contains no factors of exponent greater than 2.

It does, however, contain squares.

So the critical exponent of the Thue-Morse word is 2.
The critical exponent of a word $w$ is defined as
\[ \sup\{ r \in \mathbb{Q} : \text{w contains an } r\text{-power} \} . \]

Let $\mu$ denote the Thue-Morse morphism, defined by $\mu(0) = 01$ and $\mu(1) = 10$.

It is well-known that the Thue-Morse word $\mu^\omega(0) = 0110100110010110 \ldots$ contains no factors of exponent greater than 2.

It does, however, contain squares.

So the critical exponent of the Thue-Morse word is 2.

The repetition threshold for a set of words $L$ is the smallest critical exponent among all infinite words in $L$. 
CRITICAL EXPONENTS AND REPETITION THRESHOLDS

- The *critical exponent* of a word $w$ is defined as

$$\sup\{r \in \mathbb{Q} : w \text{ contains an } r\text{-power}\}.$$  

- Let $\mu$ denote the Thue-Morse morphism, defined by $\mu(0) = 01$ and $\mu(1) = 10$.
  - It is well-known that the Thue-Morse word
    $$\mu^\omega(0) = 0110100110010110\ldots$$
    contains no factors of exponent greater than 2.
  - It does, however, contain squares.
  - So the critical exponent of the Thue-Morse word is 2.

- The *repetition threshold* for a set of words $L$ is the smallest critical exponent among all infinite words in $L$.
  - Since every long enough binary word contains a square, the repetition threshold for the set of all binary words is 2.
Question: Are there other infinite binary words with critical exponent 2? What do they look like?

Theorem (Karhumäki and Shallit, 2004): Let \( w \) be an infinite binary word with critical exponent less than \( \frac{7}{3} \). For every \( n \geq 1 \), a suffix of \( w \) has the form \( \mu_n(w^n) \) for some infinite binary word \( w^n \). In particular, if \( w \) is an infinite binary word with critical exponent less than \( \frac{7}{3} \), then it contains every factor of the Thue-Morse word.
Question: Are there other infinite binary words with critical exponent 2? What do they look like?

Answer: It turns out that every infinite binary word with critical exponent less than 7/3 looks almost like the Thue-Morse word!
A STRUCTURE THEOREM

Question: Are there other infinite binary words with critical exponent 2? What do they look like?
Answer: It turns out that every infinite binary word with critical exponent less than $7/3$ looks almost like the Thue-Morse word!

Theorem (Karhumäki and Shallit, 2004): Let $w$ be an infinite binary word with critical exponent less than $7/3$. For every $n \geq 1$, a suffix of $w$ has the form $\mu^n(w_n)$ for some infinite binary word $w_n$. 
Question: Are there other infinite binary words with critical exponent 2? What do they look like?

Answer: It turns out that every infinite binary word with critical exponent less than \(7/3\) looks almost like the Thue-Morse word!

Theorem (Karhumäki and Shallit, 2004): Let \(w\) be an infinite binary word with critical exponent less than \(7/3\). For every \(n \geq 1\), a suffix of \(w\) has the form \(\mu^n(w_n)\) for some infinite binary word \(w_n\).

In particular, if \(w\) is an infinite binary word with critical exponent less than \(7/3\), then it contains every factor of the Thue-Morse word.
A QUICK REVIEW

▶ Every long enough binary word contains a square.
▶ The Thue-Morse word contains nothing “bigger” than a square; it has critical exponent 2.
▶ This means that the repetition threshold for the set of all binary words is 2.
▶ If an infinite binary word has critical exponent less than $\frac{7}{3}$, then it contains every factor of the Thue-Morse word.
Every long enough binary word contains a square.
A QUICK REVIEW

▶ Every long enough binary word contains a square.
▶ The Thue-Morse word contains nothing “bigger” than a square; it has critical exponent 2.
Every long enough binary word contains a square.
The Thue-Morse word contains nothing “bigger” than a square; it has critical exponent 2.
This means that the repetition threshold for the set of all binary words is 2.
A QUICK REVIEW

- Every long enough binary word contains a square.
- The Thue-Morse word contains nothing “bigger” than a square; it has critical exponent 2.
- This means that the repetition threshold for the set of all binary words is 2.
- If an infinite binary word has critical exponent less than \( \frac{7}{3} \), then it contains every factor of the Thue-Morse word.
PLAN

CRITICAL EXPONENTS AND REPETITION THRESHOLDS

RICH WORDS
A palindrome is a finite word that reads the same forwards and backwards. Examples: 1001, 01010, kayak, racecar.

Theorem (Droubay, Justin, Pirillo 2001): Every word of length $n$ contains at most $n$ distinct nonempty palindromes as factors.

A finite word of length $n$ is called rich if it contains $n$ distinct nonempty palindromes.

The word 01101 contains the palindromes 0, 1, 1, 01, 10, 11, 011, so it is rich.

The word 0120 contains only the palindromes 0, 1, 2, so it is not rich.

An infinite word is called rich if all of its finite factors are rich.
A palindrome is a finite word that reads the same forwards and backwards.

Examples: 1001, 01010, kayak, racecar

Theorem (Droubay, Justin, Pirillo 2001): Every word of length \( n \) contains at most \( n \) distinct nonempty palindromes as factors.

A finite word of length \( n \) is called rich if it contains \( n \) distinct nonempty palindromes.

The word 01101 contains the palindromes 0, 1, and 101, so it is not rich.

An infinite word is called rich if all of its finite factors are rich.
RICH WORDS

- A palindrome is a finite word that reads the same forwards and backwards.
  - Examples: 1001,
RICH WORDS

- A palindrome is a finite word that reads the same forwards and backwards.
  - Examples: 1001, 01010,
Rich words

- A palindrome is a finite word that reads the same forwards and backwards.
  - Examples: 1001, 01010, kayak,
RICH WORDS

- A palindrome is a finite word that reads the same forwards and backwards.
  - Examples: 1001, 01010, kayak, racecar
A palindrome is a finite word that reads the same forwards and backwards.

Examples: 1001, 01010, kayak, racecar

Theorem (Droubay, Justin, Pirillo 2001): Every word of length \( n \) contains at most \( n \) distinct nonempty palindromes as factors.
Rich Words

- A palindrome is a finite word that reads the same forwards and backwards.
  - Examples: 1001, 01010, kayak, racecar

Theorem (Droubay, Justin, Pirillo 2001): Every word of length $n$ contains at most $n$ distinct nonempty palindromes as factors.

- A finite word of length $n$ is called rich if it contains $n$ distinct nonempty palindromes.
Rich words

- A palindrome is a finite word that reads the same forwards and backwards.
  - Examples: 1001, 01010, kayak, racecar

Theorem (Droubay, Justin, Pirillo 2001): Every word of length $n$ contains at most $n$ distinct nonempty palindromes as factors.

- A finite word of length $n$ is called rich if it contains $n$ distinct nonempty palindromes.
  - The word 01101 contains the palindromes

- The word 0120 contains only the palindromes 0, 1, and 2, so it is not rich.

- An infinite word is called rich if all of its finite factors are rich.
Rich Words

- A palindrome is a finite word that reads the same forwards and backwards.
  - Examples: 1001, 01010, kayak, racecar

Theorem (Droubay, Justin, Pirillo 2001): Every word of length $n$ contains at most $n$ distinct nonempty palindromes as factors.

- A finite word of length $n$ is called rich if it contains $n$ distinct nonempty palindromes.
  - The word 01101 contains the palindromes 0,
Rich words

- A palindrome is a finite word that reads the same forwards and backwards.
  - Examples: 1001, 01010, kayak, racecar

Theorem (Droubay, Justin, Pirillo 2001): Every word of length $n$ contains at most $n$ distinct nonempty palindromes as factors.
- A finite word of length $n$ is called rich if it contains $n$ distinct nonempty palindromes.
  - The word 01101 contains the palindromes 0, 1,
Rich words

- A palindrome is a finite word that reads the same forwards and backwards.
  - Examples: 1001, 01010, kayak, racecar

Theorem (Droubay, Justin, Pirillo 2001): Every word of length $n$ contains at most $n$ distinct nonempty palindromes as factors.

- A finite word of length $n$ is called rich if it contains $n$ distinct nonempty palindromes.
  - The word 01101 contains the palindromes 0, 1, 11,

  - The word 0120 contains only the palindromes 0, 1, 2, so it is not rich.

- An infinite word is called rich if all of its finite factors are rich.
Rich Words

- A palindrome is a finite word that reads the same forwards and backwards.
- Examples: 1001, 01010, kayak, racecar

Theorem (Droubay, Justin, Pirillo 2001): Every word of length \( n \) contains at most \( n \) distinct nonempty palindromes as factors.

- A finite word of length \( n \) is called rich if it contains \( n \) distinct nonempty palindromes.
- The word 01101 contains the palindromes 0, 1, 11, 0110,
A palindrome is a finite word that reads the same forwards and backwards.

Examples: 1001, 01010, kayak, racecar

Theorem (Droubay, Justin, Pirillo 2001): Every word of length $n$ contains at most $n$ distinct nonempty palindromes as factors.

A finite word of length $n$ is called rich if it contains $n$ distinct nonempty palindromes.

The word 01101 contains the palindromes 0, 1, 11, 0110, and 101.
Rich words

- A palindrome is a finite word that reads the same forwards and backwards.
  - Examples: 1001, 01010, kayak, racecar

Theorem (Droubabay, Justin, Pirillo 2001): Every word of length $n$ contains at most $n$ distinct nonempty palindromes as factors.

- A finite word of length $n$ is called rich if it contains $n$ distinct nonempty palindromes.
  - The word 01101 contains the palindromes 0, 1, 11, 0110, and 101, so it is rich.
Rich Words

- A *palindrome* is a finite word that reads the same forwards and backwards.
  - Examples: 1001, 01010, kayak, racecar

Theorem (Droubay, Justin, Pirillo 2001): Every word of length $n$ contains at most $n$ distinct nonempty palindromes as factors.

- A finite word of length $n$ is called *rich* if it contains $n$ distinct nonempty palindromes.
  - The word 01101 contains the palindromes 0, 1, 11, 0110, and 101, so it is rich.
  - The word 0120 contains only the palindromes 0, 1, and 2, so it is not rich.

- An infinite word is called *rich* if all of its finite factors are rich.
Rich Words

- A palindrome is a finite word that reads the same forwards and backwards.
  - Examples: 1001, 01010, kayak, racecar

Theorem (Droubay, Justin, Pirillo 2001): Every word of length $n$ contains at most $n$ distinct nonempty palindromes as factors.

- A finite word of length $n$ is called rich if it contains $n$ distinct nonempty palindromes.
  - The word 01101 contains the palindromes 0, 1, 11, 0110, and 101, so it is rich.
  - The word 0120 contains only the palindromes 0, 1, and 2, so it is not rich.
- An infinite word is called rich if all of its finite factors are rich.
Theorem (Pelantová and Starosta, 2013): Every infinite rich word contains a square.
Repetitions in rich words

Theorem (Pelantová and Starosta, 2013): Every infinite rich word contains a square.

- This result holds over any finite alphabet.
Theorem (Pelantová and Starosta, 2013): Every infinite rich word contains a square.

- This result holds over any finite alphabet.
- So, what types of powers can be avoided by infinite rich words on $k$ letters?
Theorem (Pelantová and Starosta, 2013): Every infinite rich word contains a square.

- This result holds over any finite alphabet.
- So, what types of powers can be avoided by infinite rich words on $k$ letters?
  - Cubes?
Theorem (Pelantová and Starosta, 2013): Every infinite rich word contains a square.

▶ This result holds over any finite alphabet.
▶ So, what types of powers *can* be avoided by infinite rich words on $k$ letters?
  ▶ Cubes?
  ▶ If so, what about fractional powers between 2 and 3?
Theorem (Pelantová and Starosta, 2013): Every infinite rich word contains a square.

- This result holds over any finite alphabet.
- So, what types of powers can be avoided by infinite rich words on $k$ letters?
  - Cubes?
  - If so, what about fractional powers between 2 and 3?
  - We are asking for the repetition threshold for rich words on $k$ letters, denoted $RRT(k)$. 

Theorem (Pelantová and Starosta, 2013): Every infinite rich word contains a square.

- This result holds over any finite alphabet.
- So, what types of powers *can* be avoided by infinite rich words on $k$ letters?
  - Cubes?
  - If so, what about fractional powers between 2 and 3?
  - We are asking for the repetition threshold for rich words on $k$ letters, denoted $\text{RRT}(k)$.
  - We will determine $\text{RRT}(2)$. 
Theorem (Baranwal and Shallit, 2019): There is an infinite binary rich word with critical exponent $2 + \sqrt{2}/2$.
Theorem (Baranwal and Shallit, 2019): There is an infinite binary rich word with critical exponent $2 + \sqrt{2}/2$.

- Note: $2 + \sqrt{2}/2 \approx 2.707$. 
Theorem (Baranwal and Shallit, 2019): There is an infinite binary rich word with critical exponent $2 + \sqrt{2}/2$.

- Note: $2 + \sqrt{2}/2 \approx 2.707$.
- They conjectured that this is the smallest possible critical exponent among infinite binary rich words, i.e., that $\text{RRT}(2) = 2 + \sqrt{2}/2$. 
Theorem (Baranwal and Shallit, 2019): There is an infinite binary rich word with critical exponent $2 + \sqrt{2}/2$.

- Note: $2 + \sqrt{2}/2 \approx 2.707$.
- They conjectured that this is the smallest possible critical exponent among infinite binary rich words, i.e., that $\text{RRT}(2) = 2 + \sqrt{2}/2$.
- The irrationality of $2 + \sqrt{2}/2$ makes this hard to prove!
Theorem (Baranwal and Shallit, 2019): There is an infinite binary rich word with critical exponent $2 + \sqrt{2}/2$.

- Note: $2 + \sqrt{2}/2 \approx 2.707$.
- They conjectured that this is the smallest possible critical exponent among infinite binary rich words, i.e., that $\text{RRT}(2) = 2 + \sqrt{2}/2$.
- The irrationality of $2 + \sqrt{2}/2$ makes this hard to prove!
- Baranwal and Shallit: $\text{RRT}(2) \geq 2.7$
Define morphisms $f$ and $h$ by

$$
\begin{align*}
  f(0) &= 0 \\
  f(1) &= 01 \\
  f(2) &= 011 \\
  h(0) &= 01 \\
  h(1) &= 02 \\
  h(2) &= 022.
\end{align*}
$$
Define morphisms $f$ and $h$ by

\[
\begin{align*}
  f(0) &= 0 \\
  f(1) &= 01 \\
  f(2) &= 011 \\
  h(0) &= 01 \\
  h(1) &= 02 \\
  h(2) &= 022.
\end{align*}
\]

The infinite word $f(h^\omega(0))$ is rich and has critical exponent $2 + \sqrt{2}/2$. 

[The proof was completed using the automatic theorem proving software Walnut.]
Define morphisms $f$ and $h$ by

\[
\begin{align*}
    f(0) &= 0 \\
    f(1) &= 01 \\
    f(2) &= 011 \\
    h(0) &= 01 \\
    h(1) &= 02 \\
    h(2) &= 022.
\end{align*}
\]

The infinite word $f(h^\omega(0))$ is rich and has critical exponent $2 + \sqrt{2}/2$.

- The proof was completed using the automatic theorem proving software \textit{Walnut}.
AN IRRATIONAL REPETITION THRESHOLD?
AN IRRATIONAL REPETITION THRESHOLD?

One way to show that $\text{RRT}(2) = 2 + \sqrt{2}/2$ would be to give a structure theorem for infinite binary rich words with critical exponent less than some number close to (but larger than) $2 + \sqrt{2}/2$. 

One would hope that every infinite binary rich word with critical exponent less than $14/5$ looks like $f(h_\omega(0))$. Unfortunately, this is not the case! Fortunately, it is not much worse than this.
One way to show that $\text{RRT}(2) = 2 + \sqrt{2}/2$ would be to give a structure theorem for infinite binary rich words with critical exponent less than some number close to (but larger than) $2 + \sqrt{2}/2$.

One would hope that every infinite binary rich word with critical exponent less than $14/5$ looks like $f(h^\omega(0))$. Unfortunately, this is not the case! Fortunately, it is not much worse than this.
AN IRRATIONAL REPETITION THRESHOLD?

- One way to show that $\text{RRT}(2) = 2 + \sqrt{2}/2$ would be to give a structure theorem for infinite binary rich words with critical exponent less than some number close to (but larger than) $2 + \sqrt{2}/2$.
- One would hope that every infinite binary rich word with critical exponent less than $14/5$ looks like $f(h^\omega(0))$.
- Unfortunately, this is not the case!
One way to show that $\text{RRT}(2) = 2 + \sqrt{2}/2$ would be to give a structure theorem for infinite binary rich words with critical exponent less than some number close to (but larger than) $2 + \sqrt{2}/2$.

One would hope that every infinite binary rich word with critical exponent less than $14/5$ looks like $f(h^\omega(0))$.

Unfortunately, this is not the case!

Fortunately, it is not much worse than this.
ANOTHER STRUCTURE THEOREM

Every infinite binary rich word with critical exponent less than $14/5$ looks like either $u = f(h^\omega(0))$ or $v = f(g(h^\omega(0)))$.

\begin{align*}
f(0) &= 0 & g(0) &= 011 & h(0) &= 01 \\
f(1) &= 01 & g(1) &= 0121 & h(1) &= 02 \\
f(2) &= 011 & g(2) &= 012121 & h(2) &= 022
\end{align*}
ANOTHER STRUCTURE THEOREM

Every infinite binary rich word with critical exponent less than $14/5$ looks like either $u = f(h^\omega(0))$ or $v = f(g(h^\omega(0)))$.

\[
\begin{align*}
f(0) &= 0 & g(0) &= 011 & h(0) &= 01 \\
f(1) &= 01 & g(1) &= 0121 & h(1) &= 02 \\
f(2) &= 011 & g(2) &= 012121 & h(2) &= 022
\end{align*}
\]

Theorem (Currie, Mol, and Rampersad, 2020+): Let $w$ be an infinite rich word over the binary alphabet $\{0, 1\}$ with critical exponent less than $14/5$. For every $n \geq 1$, a suffix of $w$ has the form $f(h^n(w_n))$ or $f(g(h^n(w_n)))$ for some infinite word $w_n$ over $\{0, 1, 2\}$.
AN IRRATIONAL REPETITION THRESHOLD!

Theorem (Currie, Mol, and Rampersad, 2019+): The repetition threshold for binary rich words is $2 + \sqrt{2}/2$.

Proof:
Theorem (Currie, Mol, and Rampersad, 2019+): The repetition threshold for binary rich words is $2 + \sqrt{2}/2$.

Proof:

- If an infinite binary rich word has critical exponent less than $14/5$, then it contains all factors of $u = f(h^\omega(0))$ or all factors of $v = f(g(h^\omega(0)))$. 

- Baranwal and Shallit showed that the critical exponent of $u$ is $2 + \sqrt{2}/2$.

- So it suffices to show that $v$ has critical exponent at least $2 + \sqrt{2}/2$.

- In fact, we show that $v$ is rich, and has critical exponent exactly $2 + \sqrt{2}/2$.

- Our proof technique can also be applied to $u$, providing an alternate proof of Baranwal and Shallit’s result.
Theorem (Currie, Mol, and Rampersad, 2019+): The repetition threshold for binary rich words is $2 + \sqrt{2}/2$.

Proof:

- If an infinite binary rich word has critical exponent less than $14/5$, then it contains all factors of $u = f(h^\omega(0))$ or all factors of $v = f(g(h^\omega(0)))$.
- Baranwal and Shallit showed that the critical exponent of $u$ is $2 + \sqrt{2}/2$. 
- Our proof technique can also be applied to $u$, providing an alternate proof of Baranwal and Shallit’s result.
Theorem (Currie, Mol, and Rampersad, 2019+): The repetition threshold for binary rich words is $2 + \sqrt{2}/2$.

Proof:

- If an infinite binary rich word has critical exponent less than $14/5$, then it contains all factors of $u = f(h^\omega(0))$ or all factors of $v = f(g(h^\omega(0)))$.
- Baranwal and Shallit showed that the critical exponent of $u$ is $2 + \sqrt{2}/2$.
- So it suffices to show that $v$ has critical exponent at least $2 + \sqrt{2}/2$. 
AN IRRATIONAL REPETITION THRESHOLD!

Theorem (Currie, Mol, and Rampersad, 2019+): The repetition threshold for binary rich words is $2 + \sqrt{2}/2$.

Proof:

- If an infinite binary rich word has critical exponent less than $14/5$, then it contains all factors of $u = f(h^\omega(0))$ or all factors of $v = f(g(h^\omega(0)))$.

- Baranwal and Shallit showed that the critical exponent of $u$ is $2 + \sqrt{2}/2$.

- So it suffices to show that $v$ has critical exponent at least $2 + \sqrt{2}/2$.

- In fact, we show that $v$ is rich, and has critical exponent exactly $2 + \sqrt{2}/2$. 
AN IRRATIONAL REPETITION THRESHOLD!

Theorem (Currie, Mol, and Rampersad, 2019+): The repetition threshold for binary rich words is $2 + \sqrt{2}/2$.

Proof:

- If an infinite binary rich word has critical exponent less than $14/5$, then it contains all factors of $u = f(h^\omega(0))$ or all factors of $v = f(g(h^\omega(0)))$.
- Baranwal and Shallit showed that the critical exponent of $u$ is $2 + \sqrt{2}/2$.
- So it suffices to show that $v$ has critical exponent at least $2 + \sqrt{2}/2$.
- In fact, we show that $v$ is rich, and has critical exponent exactly $2 + \sqrt{2}/2$.
- Our proof technique can also be applied to $u$, providing an alternate proof of Baranwal and Shallit’s result.
Establishing richness

For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.

Example: $\Delta(0111001) = \ldots$

Fact: $\Delta(u)$ and $\Delta(v)$ are Sturmian words.

Thank you, Edita Pelantová!

By a theorem of Rote (1994), this means that $u$ and $v$ are complementary symmetric Rote words.

By a theorem of Blondin-Massé et al. (2011), every complementary symmetric Rote word is rich.

Therefore, both $u$ and $v$ are rich!
For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.
ESTABLISHING RICHNESS

▶ For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.
▶ e.g., $\Delta(0111001) =$
Establishing Richness

For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.

- e.g., $\Delta(0111001) =$
For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.

- e.g., $\Delta(0111001) = 1$
ESTABLISHING RICHNESS

- For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.
  - e.g., $\Delta(0111001) = 1$

Thank you, Edita Pelantová!

By a theorem of Rote (1994), this means that $u$ and $v$ are complementary symmetric Rote words.

By a theorem of Blondin-Massé et al. (2011), every complementary symmetric Rote word is rich.

Therefore, both $u$ and $v$ are rich!
ESTABLISHING RICHNESS

- For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.
  - e.g., $\Delta(0111001) = 10$

- Fact: $\Delta(u)$ and $\Delta(v)$ are Sturmian words.

  - Thank you, Edita Pelantová!

  - By a theorem of Rote (1994), this means that $u$ and $v$ are complementary symmetric Rote words.

  - By a theorem of Blondin-Massé et al. (2011), every complementary symmetric Rote word is rich.

  - Therefore, both $u$ and $v$ are rich!
For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.

- e.g., $\Delta(0111001) = 10$

Thank you, Edita Pelantová!
For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.

- e.g., $\Delta(0111001) = 100$
For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.

- e.g., $\Delta(0111001) = 100$
For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.

- e.g., $\Delta(0111001) = 1001$
For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.

- e.g., $\Delta(0111001) = 1001$
For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.

- e.g., $\Delta(0111001) = 10010$
ESTABLISHING RICHNESS

- For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.
  - e.g., $\Delta(0111001) = 10010$
For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.

- e.g., $\Delta(0111001) = 100101$
ESTABLISHING RICHNESS

For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.

- e.g., $\Delta(0111001) = 100101$

Fact: $\Delta(u)$ and $\Delta(v)$ are Sturmian words.
ESTABLISHING RICHNESS

- For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.
  - e.g., $\Delta(0111001) = 100101$
- Fact: $\Delta(u)$ and $\Delta(v)$ are Sturmian words.
  - Thank you, Edita Pelantová!
For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.
  ▶ e.g., $\Delta(0111001) = 100101$

Fact: $\Delta(u)$ and $\Delta(v)$ are Sturmian words.
  ▶ Thank you, Edita Pelantová!

By a theorem of Rote (1994), this means that $u$ and $v$ are complementary symmetric Rote words.
For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.

- e.g., $\Delta(0111001) = 100101$

Fact: $\Delta(u)$ and $\Delta(v)$ are Sturmian words.

- Thank you, Edita Pelantová!

By a theorem of Rote (1994), this means that $u$ and $v$ are complementary symmetric Rote words.

By a theorem of Blondin-Massé et al. (2011), every complementary symmetric Rote word is rich.
For a binary word $w$, let $\Delta(w)$ denote the sequence of first differences of $w$ modulo 2.

- e.g., $\Delta(0111001) = 100101$

Fact: $\Delta(u)$ and $\Delta(v)$ are Sturmian words.

- Thank you, Edita Pelantová!

By a theorem of Rote (1994), this means that $u$ and $v$ are complementary symmetric Rote words.

By a theorem of Blondin-Massé et al. (2011), every complementary symmetric Rote word is rich.

Therefore, both $u$ and $v$ are rich!
We still want to determine the critical exponent of $v$. 

Remember that $\Delta(v)$ is a Sturmian word. We can apply general results on repetitions in Sturmian words to establish the critical exponent of $v$. 

```plaintext
v = 001010010110100101001011· · ·
\Delta(v) = 01111011101110111101110· · ·
```
We still want to determine the critical exponent of $v$. To do this, we relate the repetitions in $v$ to the repetitions in $\Delta(v)$. 
We still want to determine the critical exponent of $\nu$. To do this, we relate the repetitions in $\nu$ to the repetitions in $\Delta(\nu)$.

$$
\nu = 001010010110100101001011\ldots
$$

$$
\Delta(\nu) = 01111011101110111101110\ldots
$$
We still want to determine the critical exponent of $\nu$. To do this, we relate the repetitions in $\nu$ to the repetitions in $\Delta(\nu)$.

\[ \nu = 00101001011\overline{010010100101}1 \cdots \]

\[ \Delta(\nu) = 011110111011101110111011101110 \cdots \]
Establishing the Critical Exponent

- We still want to determine the critical exponent of $\nu$.
- To do this, we relate the repetitions in $\nu$ to the repetitions in $\Delta(\nu)$.

\[
\nu = 00101001011\overline{01001}01001011 \cdots \\
\Delta(\nu) = 01111011101\overline{11011}110111010 \cdots 
\]
We still want to determine the critical exponent of $\nu$.

To do this, we relate the repetitions in $\nu$ to the repetitions in $\Delta(\nu)$.

\[
\nu = 001010010110100101001011 \cdots
\]

\[
\Delta(\nu) = 01 \underline{1110} \underline{1110} \underline{1110} \underline{111} 101110 \cdots
\]
ESTABLISHING THE CRITICAL EXPONENT

- We still want to determine the critical exponent of $\nu$.
- To do this, we relate the repetitions in $\nu$ to the repetitions in $\Delta(\nu)$.

$$\nu = 00\begin{array}{|c|c|c|c|c|c|c|}
\hline
1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 1 & 0 & 1 & 1 & 1 \\
\hline
\end{array} \ldots$$

$$\Delta(\nu) = 01\begin{array}{|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 0 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 0 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 0 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 0 & 1 & 1 & 1 \\
\hline
\end{array} \ldots$$
We still want to determine the critical exponent of $\nu$.

To do this, we relate the repetitions in $\nu$ to the repetitions in $\Delta(\nu)$.

\[
\nu = 00\underline{1010} \underline{0101} 1010 \underline{0101} 001011 \cdots \\
\Delta(\nu) = 01\underline{1110} 1110 1110 \underline{111} 101110 \cdots
\]
We still want to determine the critical exponent of $\nu$.
To do this, we relate the repetitions in $\nu$ to the repetitions in $\Delta(\nu)$.

$\nu = 00 \begin{array}{c} 1010 \end{array} 0101 1010 0101 \begin{array}{c} 001011 \end{array} \cdots$

$\Delta(\nu) = 01 \begin{array}{c} 1110 \end{array} 1110 1110 111 101110 \cdots$

Remember that $\Delta(\nu)$ is a Sturmian word.
We still want to determine the critical exponent of $\nu$. To do this, we relate the repetitions in $\nu$ to the repetitions in $\Delta(\nu)$.

\[
\nu = 00\underline{1010} \underline{0101} 1010 \underline{0101} 001011 \cdots
\]
\[
\Delta(\nu) = 01\underline{1110} \underline{1110} \underline{1110} \underline{111} 101110 \cdots
\]

Remember that $\Delta(\nu)$ is a Sturmian word. We can apply general results on repetitions in Sturmian words to establish the critical exponent of $\nu$. 

Every infinite binary rich word with critical exponent less than $14/5$ looks like either $u$ or $v$.
Summary

- Every infinite binary rich word with critical exponent less than 14/5 looks like either $u$ or $v$.
- Both $u$ and $v$ are complementary symmetric Rote words; we use this fact to prove that they are rich and have critical exponent $2 + \sqrt{2}/2$. 
Every infinite binary rich word with critical exponent less than 14/5 looks like either $u$ or $v$. Both $u$ and $v$ are complementary symmetric Rote words; we use this fact to prove that they are rich and have critical exponent $2 + \sqrt{2}/2$. We conclude that the repetition threshold for binary rich words is $2 + \sqrt{2}/2$. 

**Summary**
We have focused on binary words. What about words on $k$ letters, for $k > 2$?
We have focused on binary words. What about words on $k$ letters, for $k > 2$?

- The repetition threshold for all words on $k$ letters is given by

$$RT(k) = \begin{cases} 
7/4, & \text{if } k = 3; \\
7/5, & \text{if } k = 4; \\
k/(k-1), & \text{if } k \geq 5.
\end{cases}$$
FUTURE PROSPECTS

We have focused on binary words. What about words on $k$ letters, for $k > 2$?

- The repetition threshold for all words on $k$ letters is given by

$$RT(k) = \begin{cases} 
7/4, & \text{if } k = 3; \\
7/5, & \text{if } k = 4; \\
k/(k - 1), & \text{if } k \geq 5. 
\end{cases}$$

- Determining the repetition threshold for rich words on $k > 2$ letters remains an open problem.
We have focused on binary words. What about words on $k$ letters, for $k > 2$?  

- The repetition threshold for all words on $k$ letters is given by

$$RT(k) = \begin{cases} 
7/4, & \text{if } k = 3; \\
7/5, & \text{if } k = 4; \\
k/(k-1), & \text{if } k \geq 5.
\end{cases}$$

- Determining the repetition threshold for rich words on $k > 2$ letters remains an open problem.

- Is $RRT(k)$ rational for $k > 2$?
**FUTURE PROSPECTS**

We have focused on binary words. What about words on \( k \) letters, for \( k > 2 \)?

- The repetition threshold for all words on \( k \) letters is given by

\[
RT(k) = \begin{cases} 
7/4, & \text{if } k = 3; \\
7/5, & \text{if } k = 4; \\
k/(k - 1), & \text{if } k \geq 5.
\end{cases}
\]

- Determining the repetition threshold for rich words on \( k > 2 \) letters remains an open problem.
  - Is \( RRT(k) \) rational for \( k > 2 \)?
  - Is \( \lim_{k \to \infty} RRT(k) = 2 \)?
**More about the Structure Theorem**

\[
\begin{align*}
f(0) &= 0 & g(0) &= 011 & h(0) &= 01 \\
f(1) &= 01 & g(1) &= 0121 & h(1) &= 02 \\
f(2) &= 011 & g(2) &= 012121 & h(2) &= 022
\end{align*}
\]

Theorem (Currie, Mol, and Rampersad, 2020+): Let \( w \) be an infinite rich word over the binary alphabet \( \{0, 1\} \) with critical exponent less than \( 14/5 \). For every \( n \geq 1 \), a suffix of \( w \) has the form \( f(h^n(w_n)) \) or \( f(g(h^n(w_n))) \) for some infinite word \( w_n \) over \( \{0, 1, 2\} \).
More about the Structure Theorem

\[
\begin{align*}
  f(0) &= 0 & g(0) &= 011 & h(0) &= 01 \\
  f(1) &= 01 & g(1) &= 0121 & h(1) &= 02 \\
  f(2) &= 011 & g(2) &= 012121 & h(2) &= 022 
\end{align*}
\]

Theorem (Currie, Mol, and Rampersad, 2020+): Let \( w \) be an infinite rich word over the binary alphabet \( \{0, 1\} \) with critical exponent less than \( 14/5 \). For every \( n \geq 1 \), a suffix of \( w \) has the form \( f(h^n(w_n)) \) or \( f(g(h^n(w_n))) \) for some infinite word \( w_n \) over \( \{0, 1, 2\} \).

Idea of Proof: Suppose \( w \) is an infinite binary rich word with critical exponent less than \( 14/5 \), e.g.,

\[
w = 1001100100110110010011 \ldots
\]
More about the Structure Theorem

\[
\begin{align*}
  f(0) &= 0 & g(0) &= 011 & h(0) &= 01 \\
  f(1) &= 01 & g(1) &= 0121 & h(1) &= 02 \\
  f(2) &= 011 & g(2) &= 012121 & h(2) &= 022
\end{align*}
\]

Theorem (Currie, Mol, and Rampersad, 2020+): Let \( w \) be an infinite rich word over the binary alphabet \( \{0, 1\} \) with critical exponent less than \( 14/5 \). For every \( n \geq 1 \), a suffix of \( w \) has the form \( f(h^n(w_n)) \) or \( f(g(h^n(w_n))) \) for some infinite word \( w_n \) over \( \{0, 1, 2\} \).

Idea of Proof: Suppose \( w \) is an infinite binary rich word with critical exponent less than \( 14/5 \), e.g.,

\[
w = 1 \mid 0 \mid 011 \mid 0 \mid 01 \mid 0 \mid 011 \mid 011 \mid 0 \mid 01 \mid 0 \mid 011 \ldots
\]
MORE ABOUT THE STRUCTURE THEOREM

\begin{align*}
f(0) &= 0 & g(0) &= 011 & h(0) &= 01 \\
f(1) &= 01 & g(1) &= 0121 & h(1) &= 02 \\
f(2) &= 011 & g(2) &= 012121 & h(2) &= 022
\end{align*}

Theorem (Currie, Mol, and Rampersad, 2020+): Let \( w \) be an infinite rich word over the binary alphabet \( \{0, 1\} \) with critical exponent less than \( 14/5 \). For every \( n \geq 1 \), a suffix of \( w \) has the form \( f(h^n(w_n)) \) or \( f(g(h^n(w_n))) \) for some infinite word \( w_n \) over \( \{0, 1, 2\} \).

Idea of Proof: Suppose \( w \) is an infinite binary rich word with critical exponent less than \( 14/5 \), e.g.,

\[ w = 1 | 0 | 011 | 0 | 01 | 0 | 011 | 011 | 0 | 01 | 0 | 011 \cdots \]

So some suffix of \( w \) can be written in the form \( f(w') \).
Now consider \( w' \).
Now consider $w'$.  
  
- Do some backtracking to show that a handful of short factors cannot appear in $w'$.  
  
Obviously, the word $w'$ must be cube-free.  

So this gives us a large set of forbidden factors in $w'$.  

Divide into two cases:  

- $w'$ contains the factor $0110$.  
- $w'$ does not contain the factor $0110$.  

Now consider $w'$.

- Do some backtracking to show that a handful of short factors cannot appear in $w'$.
- Show that $w'$ must be rich.
Now consider $w'$.  
- Do some backtracking to show that a handful of short factors cannot appear in $w'$.  
- Show that $w'$ must be rich.  
- Obviously, the word $w'$ must be cube-free.
Now consider $w'$.

- Do some backtracking to show that a handful of short factors cannot appear in $w'$.
- Show that $w'$ must be rich.
- Obviously, the word $w'$ must be cube-free.
- So this gives us a large set of forbidden factors in $w'$. 
Now consider \( w' \).

- Do some backtracking to show that a handful of short factors cannot appear in \( w' \).
- Show that \( w' \) must be rich.
- Obviously, the word \( w' \) must be cube-free.
- So this gives us a large set of forbidden factors in \( w' \).
- Divide into two cases:
  - \( w' \) contains the factor 0110.
  - \( w' \) does not contain the factor 0110.
\[ f(0) = 0 \quad g(0) = 011 \quad h(0) = 01 \]
\[ f(1) = 01 \quad g(1) = 0121 \quad h(1) = 02 \]
\[ f(2) = 011 \quad g(2) = 012121 \quad h(2) = 022 \]

Case 1: \( w' \) contains the factor 0110
\[
\begin{align*}
  f(0) &= 0 & g(0) &= 011 & h(0) &= 01 \\
  f(1) &= 01 & g(1) &= 0121 & h(1) &= 02 \\
  f(2) &= 011 & g(2) &= 012121 & h(2) &= 022
\end{align*}
\]

Case 1: \( w' \) contains the factor 0110
Case 1: \( w' \) contains the factor 0110

Show that the word ending at every unboxed leaf of this tree contains a forbidden factor.
\(f(0) = 0\) \quad \(g(0) = 011\) \quad \(h(0) = 01\)

\(f(1) = 01\) \quad \(g(1) = 0121\) \quad \(h(1) = 02\)

\(f(2) = 011\) \quad \(g(2) = 012121\) \quad \(h(2) = 022\)

Case 1: \(w'\) contains the factor 0110

- Show that the word ending at every unboxed leaf of this tree contains a forbidden factor.
- So a suffix of \(w'\) can be written in the form \(f(g(w''))\).
\[
\begin{align*}
    f(0) &= 0 & g(0) &= 011 & h(0) &= 01 \\
    f(1) &= 01 & g(1) &= 0121 & h(1) &= 02 \\
    f(2) &= 011 & g(2) &= 012121 & h(2) &= 022
\end{align*}
\]

Case 1: \(w'\) contains the factor \(0110\)

▶ Show that the word ending at every unboxed leaf of this tree contains a forbidden factor.
▶ So a suffix of \(w'\) can be written in the form \(f(g(w''))\).
▶ Apply a similar argument to show that some suffix of \(w''\) can be written in the form \(f(g(h(w_1))))\).
Case 2: $w'$ does not contain the factor $0110$.

Use a similar argument to show that some suffix of $w'$ can be written in the form $f(h(w_1))$.

So altogether, we see that $w$ has a suffix of the form $f(g(h(w_1))))$, or a suffix of the form $f(h(w_1))$.

This completes the base case of an inductive proof.

The inductive step is proved by a similar (though slightly more technical) unified argument.
Case 2: $w'$ does not contain the factor $0110$

- Use a similar argument to show that some suffix of $w'$ can be written in the form $f(h(w_1))$. 

This completes the base case of an inductive proof.

The inductive step is proved by a similar (though slightly more technical) unified argument.
Case 2: $w'$ does not contain the factor $0110$

- Use a similar argument to show that some suffix of $w'$ can be written in the form $f(h(w_1))$.

- So altogether, we see that $w$ has a suffix of the form $f(g(h(w_1)))$, or a suffix of the form $f(h(w_1))$. 

This completes the base case of an inductive proof.

The inductive step is proved by a similar (though slightly more technical) unified argument.
Case 2: $w'$ does not contain the factor $0110$

- Use a similar argument to show that some suffix of $w'$ can be written in the form $f(h(w_1))$.

- So altogether, we see that $w$ has a suffix of the form $f(g(h(w_1)))$, or a suffix of the form $f(h(w_1))$.
- This completes the base case of an inductive proof.
Case 2: $w'$ does not contain the factor 0110

- Use a similar argument to show that some suffix of $w'$ can be written in the form $f(h(w_1))$.

- So altogether, we see that $w$ has a suffix of the form $f(g(h(w_1)))$, or a suffix of the form $f(h(w_1))$.
- This completes the base case of an inductive proof.
- The inductive step is proved by a similar (though slightly more technical) unified argument.
Why 14/5?

- The constant 14/5 is used in the backtracking at the beginning of the argument.
Why $14/5$?

- The constant $14/5$ is used in the backtracking at the beginning of the argument.
- In fact, it appears that the following binary words are rich and have critical exponent equal to $14/5$:

$$\tilde{f}(h^\omega(0)) \text{ and } \tilde{f}(g(h^\omega(0))),$$

where

$$\begin{align*}
\tilde{f}(0) &= 0 & g(0) &= 011 & h(0) &= 01 \\
\tilde{f}(1) &= 011 & g(1) &= 0121 & h(1) &= 02 \\
\tilde{f}(2) &= 01 & g(2) &= 012121 & h(2) &= 022
\end{align*}$$

This suggests that $14/5$ is indeed the largest possible constant for which the structure theorem holds.
WHY 14/5?

- The constant 14/5 is used in the backtracking at the beginning of the argument.
- In fact, it appears that the following binary words are rich and have critical exponent equal to 14/5:

  \[ \tilde{f}(h^\omega(0)) \text{ and } \tilde{f}(g(h^\omega(0))), \]

where

\[
\tilde{f}(0) = 0 \quad g(0) = 011 \quad h(0) = 01 \\
\tilde{f}(1) = 011 \quad g(1) = 0121 \quad h(1) = 02 \\
\tilde{f}(2) = 01 \quad g(2) = 012121 \quad h(2) = 022
\]

- This suggests that 14/5 is indeed the largest possible constant for which the structure theorem holds.
Thank you!