Maximizing the Mean Subtree Order

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**Plan**

**Background**

**The Gluing Lemma**

**Optimal Batons**

**Optimal Caterpillars**

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**Mean Subtree Order**

The *mean subtree order* of a tree $T$, denoted $M_T$, is the average number of nodes in a subtree of $T$. 

The study of the mean subtree order was initiated by Jamison in 1983. He demonstrated that among all trees of order $n$, the path has minimum mean subtree order $n + \frac{2}{3}$. The problem of characterizing those trees of order $n$ having maximum mean subtree order remains largely open. Jamison conjectured that any such tree is a *caterpillar*. 
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- Jamison conjectured that any such tree is a *caterpillar*. 
Optimal Trees

Terminology: A tree $T$ with maximum mean subtree order in a given family $\mathcal{F}$ is called optimal in $\mathcal{F}$.

Question: What do the optimal trees among all trees of order $n$ look like?

Order 4:
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Order 5:
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Order 7:
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Terminology: A tree $T$ with maximum mean subtree order in a given family $\mathcal{F}$ is called *optimal* in $\mathcal{F}$.

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**Order 8:**

![Diagram of an optimal tree with order 8]
**Optimal Trees**

Terminology: A tree $T$ with maximum mean subtree order in a given family $\mathcal{F}$ is called *optimal* in $\mathcal{F}$.

Question: What do the optimal trees among all trees of order $n$ look like?

Order 9:
Optimal Trees

Terminology: A tree $T$ with maximum mean subtree order in a given family $\mathcal{F}$ is called *optimal* in $\mathcal{F}$.

Question: What do the optimal trees among all trees of order $n$ look like?

Order 10:
OPTIMAL TREES

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Question: What do the optimal trees among all trees of order $n$ look like?

Order 11:
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Terminology: A tree $T$ with maximum mean subtree order in a given family $\mathcal{F}$ is called \textit{optimal} in $\mathcal{F}$.

Question: What do the optimal trees among all trees of order $n$ look like?

Order 12:
**Optimal Trees**

Terminology: A tree $T$ with maximum mean subtree order in a given family $\mathcal{F}$ is called *optimal* in $\mathcal{F}$.

Question: What do the optimal trees among all trees of order $n$ look like?

Order 13:
**OPTIMAL TREES**

Terminology: A tree $T$ with maximum mean subtree order in a given family $\mathcal{F}$ is called *optimal* in $\mathcal{F}$.

Question: What do the optimal trees among all trees of order $n$ look like?

Order 14:
**OPTIMAL TREES**

Terminology: A tree $T$ with maximum mean subtree order in a given family $\mathcal{F}$ is called *optimal* in $\mathcal{F}$.

Question: What do the optimal trees among all trees of order $n$ look like?

Order 15:
**Optimal Trees**

Terminology: A tree $T$ with maximum mean subtree order in a given family $\mathcal{F}$ is called *optimal* in $\mathcal{F}$.

Question: What do the optimal trees among all trees of order $n$ look like?

Order 16:
Optimal Trees

Terminology: A tree $T$ with maximum mean subtree order in a given family $\mathcal{F}$ is called optimal in $\mathcal{F}$.

Question: What do the optimal trees among all trees of order $n$ look like?

Order 17:
Optimal Trees

Terminology: A tree $T$ with maximum mean subtree order in a given family $\mathcal{F}$ is called optimal in $\mathcal{F}$.

Question: What do the optimal trees among all trees of order $n$ look like?

Order 18:
The density of a tree $T$, denoted $\text{den}(T)$, is given by

$$\text{den}(T) = \frac{M_T}{|V(T)|}.$$
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Jamison proved the following key results:

- There is a sequence $\{T_k\}$ of trees such that $\text{den}(T_k) \to 1$. 

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Jamison proved the following key results:

- There is a sequence $\{T_k\}$ of trees such that $\text{den}(T_k) \rightarrow 1$.
- For a tree $T$ of order $n \geq 3$ with $\ell$ leaves,

$$M_T \leq n - \frac{\ell}{2}.$$
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- There is a sequence $\{T_k\}$ of trees such that $\text{den}(T_k) \to 1$.
- For a tree $T$ of order $n \geq 3$ with $\ell$ leaves,

$$M_T \leq n - \frac{\ell}{2}.$$ 

- Thus, if $\text{den}(T_k) \to 1$, the proportion of vertices of $T_k$ of degree 2 approaches 1.
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**Background**

**The Gluing Lemma**

**Optimal Batons**

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**Conclusion**
THE GLUING LEMMA

Among all such trees $T$, the tree $T_{\lfloor n/2 \rfloor}$ is optimal.

In fact, if $r < s \leq \lfloor n/2 \rfloor$, then $M_{T_r} < M_{T_s}$. 

---

The diagram shows a graph $P$ with vertices $u_1, u_2, \ldots, u_n$, and a circle labeled $Q$ containing a vertex $v$. The graph $P$ is connected, and the circle $Q$ is disjoint from the path $P$. The lemma states that among all possible such graphs $T$, the graph with $\lfloor n/2 \rfloor$ is the optimal one.
THE GLUING LEMMA

Lemma
Among all such trees $T_s$, the tree $T_{\lfloor n + \frac{1}{2} \rfloor}$ is optimal.

In fact, if $r < s \leq n + \frac{1}{2}$, then $M^P_{T_r} < M^P_{T_s}$.
**THE GLUING LEMMA**

Let $P$ be a tree with vertices $u_1, u_2, \ldots, u_{n-1}, u_n$. Let $Q$ be a cycle connecting $u_s = v$ such that $u_s$ is the only vertex in $Q$ that is not in $P$. If $r < s \leq n + 1/2$, then $M_{T_r} < M_{T_s}$. Call this tree $T_s$. Among all such trees $T_s$, the tree $T_{\lfloor n + 1/2 \rfloor}$ is optimal.
THE GLUING LEMMA

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In fact, if $r < s \leq \frac{n+1}{2}$, then $M_T(r) < M_T(s)$.

- Call this tree $T_s$. 

![Diagram](image-url)
THE GLUING LEMMA

Call this tree $T_s$.

Lemma

Among all such trees $T_s$, the tree $T_{\left\lfloor \frac{n+1}{2} \right\rfloor}$ is optimal.
The Gluing Lemma

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In fact, if $r < s \leq \frac{n+1}{2}$, then $M_{Tr} < M_{Ts}$. 

Call this tree $T_s$. 

Lemma
LIMBS OF OPTIMAL TREES

A limb of $T$ is a maximal path in $T$ containing a leaf of $T$ and no vertices of degree greater than 2 in $T$. 
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![Diagram of limbs of an optimal tree]

**Theorem**

If $T$ is optimal among all trees of order $n$, then every limb of $T$ has order 1.
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![Diagram showing limbs of an optimal tree]

**Theorem**

*If $T$ is optimal among all trees of order $n$, then every limb of $T$ has order 1.*
The Gluing Lemma allows us to find optimal trees in certain restricted families. Finally, the Gluing Lemma helps us to give a positive answer to a question of Jamison: if $T$ is not a path, then there is a 1-associate $T'$ of $T$ such that $M_T < M_{T'}$.Jamison had proven this with the extra condition that $M_T \leq n + 1/2$. 

**Other Implications of the Gluing Lemma**
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Let $s, t, n \in \mathbb{N}$ such that $s + t \leq n - 2$. Let $B_n(s, t)$ denote the tree of order $n$ obtained from the stars $K_{1,s}$ and $K_{1,t}$ by joining the centre vertices with a path of length $n - s - t - 2$. 
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Questions:
▶ Are the optimal batons balanced?
▶ How many leaves do the optimal batons have?
Batons

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Questions:
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Theorem

Among all batons on $n$ vertices with $2s$ leaves, the balanced baton $B_n(s, s)$ is optimal whenever $s \geq \log_2(n)$.
OPTIMAL BATONS

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Theorem

If \( B_n(s_n, s_n) \) is optimal among all balanced batons of order \( n \), then for \( n \) sufficiently large,

\[
2 \log_2(n) - 2 < s_n < 2 \log_2(n) + 1.
\]
OPTIMAL BATONS

Theorem

Among all batons on n vertices with 2s leaves, the balanced baton \( B_n(s, s) \) is optimal whenever \( s \geq \log_2(n) \).

Theorem

If \( B_n(s_n, s_n) \) is optimal among all balanced batons of order n, then for n sufficiently large,

\[
2 \log_2(n) - 2 < s_n < 2 \log_2(n) + 1.
\]

Corollary

For each natural number n, there is a caterpillar of order n with mean subtree order at least \( n - 2 \log_2(n) - 2 \).
This last corollary, together with Jamison’s bound on the mean subtree order in terms of the number of leaves, tells us something about optimal trees among all trees of a fixed order.
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Corollary

Suppose that $T_n$ is an optimal tree among all trees of order $n$. Then $T_n$ has at most

$$4 \log_2(n) + 4$$

leaves.
**Plan**

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**Conclusion**
THE NUMBER OF LEAVES IN AN OPTIMAL CATERPILLAR

- It would be nice to get a lower bound on the number of leaves in an optimal tree among all trees of order $n$. 

Theorem
If $T$ is optimal among all caterpillars of order $n$, then $T$ has at least $\log_2(n) - \log_2(\log_2(n) + 1) - \log_2(3)$ leaves.

It follows that the number of leaves in an optimal caterpillar of order $n$ is $\Theta(\log_2(n))$. 
The number of leaves in an optimal caterpillar

- It would be nice to get a lower bound on the number of leaves in an optimal tree among all trees of order $n$.
- We have achieved a lower bound on the number of leaves in an optimal tree among all caterpillars of order $n$. 

$\text{Theorem}$

If $T$ is optimal among all caterpillars of order $n$, then $T$ has at least

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**Theorem**

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- We have achieved a lower bound on the number of leaves in an optimal tree among all caterpillars of order \( n \).

Theorem

If \( T \) is optimal among all caterpillars of order \( n \), then \( T \) has at least

\[ \log_2(n) - \log_2(\log_2(n) + 1) - \log_2(3), \]

leaves.

- It follows that the number of leaves in an optimal caterpillar of order \( n \) is \( \Theta(\log_2(n)) \).
PROOF

The tree obtained from $T$ by deleting all leaves is called the \textit{stem} of $T$. 
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**Lemma**

\begin{align*}
\text{Let } T \text{ be a tree with } \ell \leq n - 2 \text{ leaves and let } S \text{ be the stem of } T. \text{ Then} \\
N_T & \leq N_S \cdot 2^\ell,
\end{align*}

where $N_T$ is the number of subtrees of $T$ and $N_S$ is the number of subtrees of $S$. 
PROOF

- Let $T$ be optimal among all caterpillars of order $n$. 
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- Let $T$ be optimal among all caterpillars of order $n$.
- Suppose that $T$ has $\ell \leq \log_2 \left( \frac{n}{3 \log_2(n) + 3} \right)$ leaves.
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- Let $T$ be optimal among all caterpillars of order $n$.
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▷ The subtrees of $T$ can be partitioned into two types:
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  1. Those that are contained in $S$. 
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  1. Those that are contained in $S$.
     - There are $N_S$ such subtrees with mean order $M_S$. 
PROOF

- Let $T$ be optimal among all caterpillars of order $n$.
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   - There are $N_S$ such subtrees with mean order $M_S$.
   - Since $S$ is a path of order at most $n - 2$, $M_S \leq \frac{n}{3}$. 
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  2. Those that are not contained in $S$.
     - There are $N_T - N_S$ such subtrees and we denote their mean order $\overline{M}_S$. 
PROOF

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- We want to show that $M_T < n - 2 \log_2(n) - 2$. 
## PLAN

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### Key results

1. In any optimal tree of order $n$, all limbs have order 1.
2. An optimal tree of order $n$ has $O(\log_2 n)$ leaves.
3. An optimal caterpillar of order $n$ has $\Theta(\log_2 n)$ leaves.
Key Results

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- An optimal tree of order $n$ has $O(\log_2 n)$ leaves.

- An optimal caterpillar of order $n$ has $\Theta(\log_2 n)$ leaves.
OPEN PROBLEMS

- While the caterpillar conjecture remains undecided, we suspect that it is true. We have tried several proof techniques to no avail.

- While we have an upper bound on the number of leaves in an optimal tree of order $n$ (and we suspect that it is fairly tight), we lack a lower bound.

- We suspect that our lower bound on the number of leaves in an optimal caterpillar of order $n$ can be improved.