Circular repetition thresholds for small alphabets: 
Last cases of Gorbunova’s Conjecture

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Joint work with James D. Currie and Narad Rampersad

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Plan

Background

Four letters

Five letters

Conclusion
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Words

- A (linear) word is a finite string of letters taken from a finite alphabet.
- We could take the English alphabet, the binary alphabet \{0, 1\}, etc.
- For words \(x\) and \(y\), \(xy\) denotes the concatenation of \(x\) and \(y\).
  - e.g. If \(x = \text{book}\) and \(y = \text{case}\), then \(xy = \text{bookcase}\).
- A word \(y\) is a factor of a word \(w\) if we can write \(w = xyz\) for some (possibly empty) words \(x\) and \(z\).
  - e.g. The word \(\text{Brandon}\) has factors including \(\text{ran}\) and \(\text{Brand}\) (and \(\text{and}\)).
- The length of word \(w\) is denoted \(|w|\).
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### Repetitions

Let \( w = w_1 \ldots w_n \), where the \( w_i \) are letters.

- We say that \( w \) is periodic if for some positive integer \( p \), 
  \[ w_i + p = w_i \]
  for all \( 1 \leq i \leq n - p \).

- In this case, \( p \) is called a period of \( w \).

- e.g. The English word alfalfa has period 3.

- The exponent of \( w \) is the ratio between its length and its minimal period.

- e.g. The word alfalfa has length 7 and minimal period 3, so it has exponent \( 7/3 \).

- We will mostly be interested in factors of exponent \( \beta \) where \( 1 < \beta < 2 \), which can always be written as \( xyx \) with 
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  - The word *mathematics* has the factor *mathemat*, which has exponent $\frac{8}{5}$. However, *mathematics* is $\frac{8^+}{5}$-free.

**Theorem (Thue, 1906)**

Over a two letter alphabet, there is a word of every length that is $2^+$-free.
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Theorem (Thue, 1906)

Over a two letter alphabet, there is a word of every length that is $2^+$-free.

Further, every sufficiently long word on two letters contains a factor of exponent 2 (a square).
Dejean’s Conjecture

Definition (Dejean, 1972)

Let $k \geq 2$. The repetition threshold for $k$ letters, denoted $RT(k)$, is the infimum of the set of all $\beta$ such that there are $\beta$-free words of every length on $k$ letters.

- With this terminology, Thue demonstrated that $RT(2) = 2$.

Conjecture (Dejean, 1972)

$$
RT(k) = \begin{cases} 
7 & \text{if } k = 3 \\
5 & \text{if } k = 4 \\
k - 1 & \text{if } k \geq 5
\end{cases}
$$
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Conjecture (Dejean, 1972)

$$RT(k) = \begin{cases} \frac{7}{4} & \text{if } k = 3 \\ \frac{7}{5} & \text{if } k = 4 \\ \frac{k}{k-1} & \text{if } k \geq 5. \end{cases}$$
## Progress on Dejean’s Conjecture

\[ k = 3 \quad \text{Dejean} \quad 1972 \]
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Circular words

Intuitively, a circular word is obtained from a linear word by linking the ends, giving a cyclic sequence of letters. Factors don't "wrap around" more than once. i.e. The longest factors of a circular word of length $n$ have length $n$. As a linear word, onion is $2$-free. However, the circular word (onion) has factor onon, so it is not $2$-free.
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Circular Repetition Threshold

**Definition**

Let $k \geq 2$. The *circular repetition threshold* for $k$ letters, denoted $\text{CRT}(k)$, is the infimum of the set of all $\beta$ such that there are $\beta$-free circular words of every length on $k$ letters.
Known values of the circular repetition threshold

- CRT(2) = 5 (Aberkane, Currie, 2004)
- CRT(3) = 2 (Currie, 2002)
- Conjecture (Gorbunova, 2012)
  For all \( k \geq 4 \), CRT(\( k \)) = \( \lceil k/2 \rceil \) + 1
- Gorbunova confirmed her conjecture for all \( k \geq 6 \).
- Last remaining cases: CRT(4) and CRT(5).
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**Conjecture (Gorbunova, 2012)**

For all $k \geq 4$,

$$\text{CRT}(k) = \frac{\lceil k/2 \rceil + 1}{\lceil k/2 \rceil}$$
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# Last Cases of Gorbunova’s Conjecture

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The lower bound

Proposition (Gorbunova, 2012)

For any \( k \geq 4 \), there are no circular \( \lceil k/2 \rceil + 1 \lceil k/2 \rceil \)-free words of length \( k + 1 \) over a \( k \) letter alphabet.

Sketch of Proof.

• Pigeonhole principle.

So to prove the last two cases of Gorbunova's Conjecture, it suffices to find

• \( 3^2 + \) -free circular words of every length on 4 letters, and
• \( 4^3 + \) -free circular words of every length on 5 letters.
The lower bound

Proposition (Gorbunova, 2012)

For any $k \geq 4$, there are no circular $\lceil k/2 \rceil + 1$-free words of length $k + 1$ over a $k$-letter alphabet.
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Proposition (Gorbunova, 2012)

For any $k \geq 4$, there are no circular $\left\lceil \frac{k}{2} \right\rceil + 1$-free words of length $k + 1$ over a $k$ letter alphabet.

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**Proposition (Gorbunova, 2012)**

For any $k \geq 4$, there are no circular $\frac{\lceil k/2 \rceil + 1}{\lceil k/2 \rceil}$-free words of length $k + 1$ over a $k$-letter alphabet.

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So to prove the last two cases of Gorbunova’s Conjecture, it suffices to find
  - $\frac{3}{2}^+$-free circular words of every length on 4 letters, and
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• $\frac{4}{3}^+$-free circular words of every length on 5 letters.
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Conclusion
Morphisms

• An $r$-uniform morphism takes a word as input and replaces every letter by a word of length $r$.

• A morphism $f$ preserves $\beta$-freeness if $f(w)$ is $\beta$-free whenever $w$ is $\beta$-free.

• Iterating gives $\beta$-free words of arbitrarily long length.
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Four letters

Idea:

Four letters

Problem:
• If a linear word is $\beta$-free, then so are all of its factors.
• This is not the case for circular words.
  • e.g. $(\text{discrete})$ is $2$-free, but $(\text{ete})$ is not.

Solution:
• Use two different morphisms: an $r$-uniform morphism and an $s$-uniform morphism (where $r$ and $s$ are relatively prime).
Four letters

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• Find a uniform morphism that preserves $\frac{3}{2}^+$-freeness for circular words.
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• Starting with a single letter, and iteratively applying an $r$-uniform morphism only gives words of length $r^n$.

Solution:

• Use two different morphisms: an $r$-uniform morphism and an $s$-uniform morphism (where $r$ and $s$ are relatively prime).
Constructing circular $\frac{3}{2}^+$-free words on four letters
Constructing circular $\frac{3^+}{2}$-free words on four letters

- Find a 9-uniform morphism $f_9$ and an 11-uniform morphism $f_{11}$ that preserve $\frac{3^+}{2}$-freeness.
Constructing circular $\frac{3^+}{2}$-free words on four letters

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- Define $f_9$ by:

  \begin{align*}
  0 & \mapsto 012132310 \\
  1 & \mapsto 123203021 \\
  2 & \mapsto 230310132 \\
  3 & \mapsto 301021203
  \end{align*}
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Constructing circular $\frac{3}{2}^+$-free words on four letters
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Constructing circular $\frac{3}{2}^+$-free words on four letters

- Use a strong inductive argument.
- Find some short words by computer search to get things started.
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\[ n = 9k + 11\ell, \]

for $k \geq 8$ and $2 \leq \ell \leq 10$. 
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• Take a $\frac{3}{2}^+$-free circular word ($w$) of length $k + \ell$, and write it as $w = uv$, where $|u| = k$ and $|v| = \ell$. 
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- Take a $\frac{3}{2}^+$-free circular word $(w)$ of length $k + \ell$, and write it as $w = uv$, where $|u| = k$ and $|v| = \ell$.
- Claim: $(f_9(u)f_{11}(v))$ is $\frac{3}{2}^+$-free.
Constructing circular $\frac{3}{2}^+$-free words on four letters

Sketch of Proof.

Suppose otherwise that $(f_9(u)f_{11}(v))$ contains some factor with exponent greater than $\frac{3}{2}$. 
Constructing circular $\frac{3}{2}^+$-free words on four letters

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Suppose otherwise that $(f_9(u) f_{11}(v))$ contains some factor with exponent greater than $\frac{3}{2}$.

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Constructing circular $\frac{3}{2}^+$-free words on four letters

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- Then $(f_9(u)f_{11}(v))$ has some factor of the form $xyx$, where $|x| > |y|$.
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```
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  |
  x y x
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\[\begin{array}{c|c|c}
 & f_9(u) & f_{11}(v) \\
\hline
x & y & x
\end{array}\]
Constructing circular $\frac{3}{2}^+\text{-free}$ words on four letters

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Therefore,
Therefore, 

\[ \text{CRT}(4) = \frac{3}{2}. \]
Plan

Background

Four letters

Five letters

Conclusion
Gorbunova’s technique for larger alphabets

- e.g. seven letter alphabet \{0, 1, 2, 3, 4, 5, 6\}.

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Gorbunova’s technique for larger alphabets
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• Let $u$ be a $\frac{5}{4}^+$-free word on $\{0, 1, 2, 3, 4\}$. 
Gorbunova’s technique for larger alphabets

- Let $u$ be a $\frac{5}{4}^+$-free word on $\{0, 1, 2, 3, 4\}$.
- Let

\[
w = \begin{array}{ccc}
06 & u & 60 \\
\{0, 1, 2, 3, 4\} & f(u) & \{2, 3, 4, 5, 6\}
\end{array}
\]

where $f$ is a particular bijection.
Gorbunova’s technique for larger alphabets

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where $f$ is a particular bijection.
- Claim: There is such a word $u$ of every length so that $(w)$ is $\frac{5}{4}$-free.
Constructing circular $\frac{4}{3}^+$-free words on five letters
Constructing circular $\frac{4}{3}^+$-free words on five letters

Idea:
Constructing circular $\frac{4}{3}^+$-free words on five letters

Idea:
- Get a $\frac{4}{3}^+$-free word on four letters and use a construction similar to Gorbunova’s.

$$w = \begin{array}{c|c|c}
04 & u & 40 \\
\{0, 1, 2, 3\} & & \{1, 2, 3, 4\} \\
\end{array}$$

$$f(u)$$
Constructing circular $\frac{4}{3}^+\text{-free}$ words on five letters

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- 7/5 is larger than 4/3. Uh-oh.
Constructing circular $\frac{4+}{3}$-free words on five letters
Constructing circular $\frac{4}{3}^+$-free words on five letters

Solution:
Constructing circular $\frac{4}{3}^+$-free words on five letters

Solution:

• In his work showing that $RT(4) = \frac{7}{5}$, Pansiot constructed words of every length where the only factors of exponent greater than $\frac{4}{3}$ look like

$$0123\ 102132\ 0123$$

up to permutation of the letters.
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  - Fact: We don’t need the fifth letter too often.
  
  $$w = \begin{array}{|c|c|c|}
  \hline
  04 & u & 40 \\
  \hline
  \{0,1,2,3\} & \{1,2,3,4\} \\
  \text{(and a few 4’s)} & \text{(and a few 0’s)} \\
  \hline
\end{array}$$
Therefore,
Therefore, 

\[ \text{CRT}(5) = \frac{4}{3}. \]
Plan

Background

Four letters

Five letters

Conclusion
We now know that

\[
\text{CRT}(k) = \begin{cases} 
\frac{5}{2} & \text{if } k = 2; \\
2 & \text{if } k = 3; \\
\left\lceil \frac{k/2}{\lfloor k/2 \rfloor} \right\rceil + 1 & \text{if } k \geq 4.
\end{cases}
\]
Something to think about...

**Conjecture**

For all $k \geq 4$,

$$\text{CRT}_W(k) = \text{CRT}_I(k) = \text{RT}(k).$$
Thank you.