GLOBAL CRYSTAL BASES AND $q$-SCHUR ALGEBRAS

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Abstract. We prove that the quantized Carter-Lusztig basis for a finite dimensional irreducible $U_q(\mathfrak{gl}_n(\mathbb{C}))$-module $V(\lambda)$ is related to the global crystal basis for $V(\lambda)$ by an upper triangular invertible matrix. We express the global crystal basis in terms of the $q$-Schur algebra and provide an algorithm for obtaining global crystal basis vectors for $V(\lambda)$ using the $q$-Schur algebra.

1. Introduction

Various bases for finite dimensional irreducible polynomial representations of the quantized universal enveloping algebra $U_q(\mathfrak{gl}_n(\mathbb{C}))$ have been given. Each such $U_q(\mathfrak{gl}_n)$-module is of the form $V(\lambda)$, where $\lambda$ is a partition of a positive integer into at most $n$ parts, and the dimension of $V(\lambda)$ is given by the number of semistandard $\lambda$-tableaux with entries in the set $\{1, 2, \ldots, n\}$. Several authors have studied transition matrices between various bases (see, for instance, [2], [4], [13]).

The canonical bases or global crystal bases of $V(\lambda)$ due to Lusztig [14] and Kashiwara [12] have nice properties but can be difficult to compute explicitly. Algorithms to compute global crystal basis vectors are given by de Graaf in [4] and Leclerc-Toffin in [13]. By embedding $V(\lambda)$ into a tensor product of fundamental modules, Leclerc and Toffin give an intermediate monomial basis for $V(\lambda)$ which is shown to be related to the global crystal basis of $V(\lambda)$ by a unitriangular matrix. They then obtain the global crystal basis vectors through a triangular algorithm.

Polynomial representations of $U_q(\mathfrak{gl}_n)$ can also be studied by means of the $q$-Schur algebra, $S_q(n,r)$. This is a quantized version of the classical Schur algebra $S(n,r)$ which was defined by J. A. Green [7] as the dual of the coalgebra $A(n,r)$ of homogeneous polynomials of degree $r$ in $n^2$ variables $x_{ij}$, $1 \leq i, j \leq n$. There are several different approaches to studying $q$-Schur algebras in the literature (see [1], [5], [6], [17]). We follow the approach taken by J. A. Green, but in the quantum setting (see [17]), where $A_q(n)$ is the coordinate ring of quantum matrices, due to Manin [15], $A_q(n,r)$ is the $r$th homogeneous part of $A_q(n)$, and $S_q(n,r)$ is the dual $A_q(n,r)^*$.

A quantized version of the Carter-Lusztig basis for $V(\lambda)$, given in terms of elements in $U_q(\mathfrak{gl}_n)^+$, is given in [18]. In [3], we give the Carter-Lusztig basis in

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terms of $q$-Schur algebra elements. The primary aims of the current work are to describe the global crystal basis in terms of elements in the $q$-Schur algebra, to give an algorithm that explicitly provides elements of the global crystal basis using $q$-Schur algebra elements, and to prove that the Carter-Lusztig basis and global crystal basis are related by an invertible, upper triangular matrix.

After recalling the necessary background material, we discuss Leclerc-Toffin’s intermediate basis in Section 4. We then develop various results regarding $q$-Schur algebras that allow us to explicitly prove at the end of Section 6 that the transition matrix between the quantized Carter-Lusztig basis and the Leclerc-Toffin intermediate basis is upper triangular and invertible, from which it follows that the Carter-Lusztig basis and global crystal basis are related by an invertible, upper triangular, matrix. We give a method for determining the entries of the first column of the transition matrix between the quantized Carter-Lusztig basis and the Leclerc-Schur algebras that allow us to explicitly prove at the end of Section 6 that the Carter-Lusztig basis and global crystal basis are related by an invertible, upper triangular matrix.

2. Young tableaux

Let $n$ and $r$ be fixed positive integers and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$ and $\sum_{i=1}^k \lambda_i = r$, be a partition of $r$, denoted $\lambda \vdash r$. Define $\Lambda^+(n, r) = \{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash r \mid k \leq n \}$ and $I(n, r) = \{ I = (i_1, i_2, \ldots, i_r) \mid i_\rho \in \{1, \ldots, n\}, 1 \leq \rho \leq r \}$.

All partitions $\lambda$ shall belong to $\Lambda^+(n, r)$. The Young diagram of shape $\lambda$ consists of $k$ left-justified rows where the $i$-th row contains $\lambda_i$ boxes and a $\lambda$-tableau is a filling of the Young diagram of shape $\lambda$ with entries from $\{1, 2, \ldots, n\}$.

A $\lambda$-tableau is semistandard if it is both column increasing and weakly row increasing. Denote the set of $\lambda$-tableau by $T(\lambda, n)$ and let $CT(\lambda, n) = \{ T \in T(\lambda, n) \mid T \text{ is column increasing} \}$, $RT(\lambda, n) = \{ T \in T(\lambda, n) \mid T \text{ is weakly row increasing} \}$, $SST(\lambda, n) = \{ T \in T(\lambda, n) \mid T \text{ is semistandard} \}$.

The column sequence $I_C(T)$ of $T$ comes from reading the entries down columns from left to right and the row sequence $I_R(T)$ from reading the entries across the rows of $T$ from top to bottom. If $I = I_R(T)$ is the row sequence of $T$, we will often write $I^t$ to denote the corresponding column sequence $I_C(T)$ of $T$.

We will often work with the column and row sequences of the tableau $T(\lambda)$, which is obtained by filling the $i$-th row of the Young diagram of shape $\lambda$ entirely with $i$’s. Denote $I_R(T(\lambda)) = I(\lambda)$ and $I_C(T(\lambda)) = I_C(\lambda)$.

The symmetric group acts on $I(n, r)$ by $I\sigma = (i_1, \ldots, i_r)\sigma = (i_{\sigma(1)}, \ldots, i_{\sigma(r)})$, for $\sigma \in S_r$, which yields an action on $\lambda$-tableaux by defining $T\sigma = S$ where $I_C(S) = I_C(T)\sigma$. Let $T^\lambda$ be the $\lambda$-tableau with row sequence $I_R(T^\lambda) = (1, 2, \ldots, r)$ and define $C(\lambda)$ to be the subgroup of permutations in $S_r$ that leave the columns of $T^\lambda$ invariant and $R(\lambda)$ the subgroup that leaves the rows of $T^\lambda$ invariant. Two $\lambda$-tableaux $T$ and $S$ are row equivalent if $T = S\sigma$ for some $\sigma \in R(\lambda)$; we denote
We will need to make minor adjustments to some of the required
Remark 1.

Another comultiplication \( \Delta \) is denoted by
\( \Delta_1 : U_q(\mathfrak{gl}_n) \to U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_n) \) is given by \( \Delta_1 = \tau \otimes \tau \circ \Delta \circ \tau \). We then have the following:

(2) \( \Delta_1(E_i) = 1 \otimes E_i + E_i \otimes \Delta_1(E_i), \Delta_1(F_i) = K_{i,i+1}^{-1} \otimes F_i + F_i \otimes 1, \Delta_1(K_j) = K_j \otimes K_j, \)
where \( 1 \leq i < n, 1 \leq j \leq n. \)

Remark 1. We will need to make minor adjustments to some of the required
results from [3] and [11] since the comultiplication \( \Delta \) was used in those articles.
For a $U_q(\mathfrak{gl}_n)$-module $V$ and $\chi = (\chi_1, \ldots, \chi_n)$ an $n$-tuple of non-negative integers, the \textit{weight space} associated to $\chi$ is the subspace $V^\chi = \{v \in V \mid K_i v = q^{\chi_i} v, \ 1 \leq i \leq n\}$. If $v \in V^\chi$, $v \neq 0$, then $v$ is said to be a \textit{weight vector} of weight $\chi$, and $v$ is a \textit{highest-weight vector} if $E_i v = 0$ for $1 \leq i < n$.

Let $A_q(n)$ be the associative $\mathbb{C}(q)$-algebra generated by the variables $x_{ij}$, $1 \leq i, j \leq n$, subject to the relations (see [15], [19], for instance):

\begin{align}
\sum_{1 \leq k < l \leq n} x_{ik} x_{kl} &= q x_{il} x_{lk} \quad 1 \leq k < l \leq n \\
\sum_{1 \leq i < j \leq n} x_{ij} x_{jk} &= x_{ik} x_{jk} \quad 1 \leq i < j \leq n, 1 \leq k < l \leq n \\
\sum_{1 \leq i < j \leq n} x_{ij} x_{ik} &= (q^{-1} - q) x_{ij} x_{ik} \quad 1 \leq i < j \leq n, 1 \leq k < l \leq n.
\end{align}

Given $I = (i_1, \ldots, i_r)$, $J = (j_1, \ldots, j_r) \in I(n, r)$, let $x_{I, J} = x_{i_1 j_1} \cdots x_{i_r j_r} \in A_q(n)$ and let $A_q(n, r)$ denote the $\mathbb{C}(q)$-subspace of $A_q(n)$ generated by the monomials $x_{I, J}$, where $I, J \in I(n, r)$. The algebra $A_q(n)$ is a coalgebra, with comultiplication given by $\Delta(x_{ij}) = \sum_{k=1}^n x_{ik} \otimes x_{kj}$, and $A_q(n, r)$ is a subcoalgebra of $A_q(n)$. The dual $A_q(n, r)^* = \mathbb{C}(q, n, r)$, is then an associative $\mathbb{C}(q)$-algebra called the \textit{q-Schur algebra} with multiplication $\xi \eta(x_{I, J}) = \sum_{A \in I(n, r)} \xi(x_{I, A}) \eta(x_{A, J})$, where $\xi, \eta \in \mathbb{C}(q, n, r)$, $x_{I, J} \in A_q(n, r)$.

Let $I(n, r)^2 = I(n, r) \times I(n, r)$ and define

$$J(n, r) = \{(I, J) \in I(n, r)^2 \mid j_1 \leq j_2 \leq \cdots \leq j_r \text{ and } i_k \leq i_{k+1} \text{ when } j_k = j_{k+1}\}.$$ Then $\{x_{I, J} \mid (I, J) \in J(n, r)\}$ is a basis for $A_q(n, r)$ (see [5]). We will often shorten the notation for $J(n, r)$ to $J$.

The dual basis $\{\xi_{I, J} \mid (I, J) \in J(n, r)\}$ for $S_q(n, r)$ satisfies $\xi_t I, J(x_{P, Q}) = 1$ if $x_{P, Q} = x_{I, J}$ and $\xi_t I, J(x_{P, Q}) = 0$ otherwise, where $(P, Q), (I, J) \in J(n, r)$. For arbitrary $(I, J) \in I(n, r)^2$, we define

$$\xi_{I, J} = \sum_{(A, B) \in J} c_{A, B} \xi_{A, B} \quad \text{where} \quad x_{I, J} = \sum_{(A, B) \in J} c_{A, B} x_{A, B}.$$

The symmetric group acts on $I(n, r) \times I(n, r)$ by $(I, J) \sigma = (I \sigma, J \sigma)$. Let $< \leq$ be the lexicographic order on $I(n, r)$ and order $I(n, r) \times I(n, r)$ by defining $(A, B) < (I, J)$ if $B < J$ or $B = J$ and $A < I$. Let $(I, J)_0$ be the minimal element in the $S_r$-orbit containing $(I, J)$.

For $I = (i_1, i_2, \ldots, i_r)$, $J = (j_1, j_2, \ldots, j_r) \in I(n, r)$, let $S_1 = \{(a, b) \mid a < b, \ i_a = i_b \text{ and } j_a > j_b\}$, $S_2 = \{(a, b) \mid a < b, \ j_a = j_b \text{ and } i_a > i_b\}$, and define $e(I, J) = |S_1| + |S_2|$. The following two lemmas, the first of which is an adjustment of [16, Lemma 6.1.2], will be useful throughout the article.

\textbf{Lemma 3.1.} Let $I, J \in I(n, r)$. Then

$$x_{I, J} = q^{e(I, J)} x_{(I, J)_0} + \sum_{(S, T) \in J} a_{S, T} x_{S, T},$$

where $a_{S, T} \in \mathbb{Z}[q, q^{-1}]$.

Define $\mathcal{R}(\lambda, n) = \{Q \in I(n, r) \mid Q = I_R(T) \text{ for some } T \in RT(\lambda, n)\} = \{Q \in I(n, r) \mid (Q, I(\lambda)) \in J\}.$
Lemma 3.2. Let $\eta \in S_q(n, r)$ and $\lambda \in \Lambda^+(n, r)$. Then $\eta_{I(\lambda), I(\lambda)} = \sum_{Q \in \mathcal{R}(\Lambda, n)} a_Q \xi_{Q, I(\lambda)}$, where $a_Q \in \mathbb{C}(q)$.

Proof. Write $\eta_{I(\lambda), I(\lambda)} = \sum_{(Q, P) \in \mathcal{J}} a_{Q, P} \xi_{Q, P}$ as a $\mathbb{C}(q)$-linear combination of basis elements. Then $a_{Q, P} = \eta_{I(\lambda), I(\lambda)}(x_{Q, P}) = \sum_{A \in I(n, r)} \eta(x_{Q, A}) \xi_{I(\lambda), I(\lambda)}(x_{A, P})$. But $\xi_{I(\lambda), I(\lambda)}(x_{A, P}) = 0$ unless $P \sim I(\lambda)$ and since $(Q, P) \in \mathcal{J}$, we must have $P = I(\lambda)$. Thus $\eta_{I(\lambda), I(\lambda)} = \sum_{(Q, I(\lambda)) \in \mathcal{J}} a_Q \xi_{Q, I(\lambda)}$, and $(Q, I(\lambda)) \in \mathcal{J}$ if and only if the tableau with row sequence $Q$ is weakly row increasing. \qed

4. LECLERC-TOFFIN BASES AND GLOBAL CRYSTAL BASES

We review the relevant results on $U_q(\mathfrak{gl}_n)$-modules and global bases, for the most part following [13]. We have a $U_q(\mathfrak{gl}_n)$-module action on $A_q(n)$ given by

$$E_i(PQ) = (E_i P) Q + (K_{i+1}^{-1} P)(E_i Q), \quad F_i(PQ) = (F_i P)(K_i Q) + P(F_i Q),$$

$$K_i(PQ) = (K_i P)(K_i Q), \quad P, Q \in A_q(n).$$

Given $I = (i_1, i_2, \ldots, i_r)$, $J = (j_1, j_2, \ldots, j_r) \in I(n, r)$ with $i_1 < i_2 < \cdots < i_r$, define the $q$-determinant in $A_q(n, r)$ by

$$\det_q X^I_J = \begin{cases} \sum_{\sigma \in S_r} (-q)^{-\ell(\sigma)} x_{i_1 j_1(1)} x_{i_2 j_2(2)} \cdots x_{i_r j_r(r)} & \text{if } j_1 < j_2 < \cdots < j_r, \\ \sum_{\sigma \in S_r} (-q)^{-\ell(\sigma)} x_{i_1(1) j_1} x_{i_2(2) j_2} \cdots x_{i_r(r) j_r} & \text{otherwise.} \end{cases}$$

For $k \leq n$, let $\Lambda_k = (1, 1, \ldots, 1, 0, \ldots, 0)$ and let $T$ be a $\Lambda_k$-tableau with column sequence $I^C(T) = (a_1, a_2, \ldots, a_k)$ where $a_i \in \{1, \ldots, n\}$ for $1 \leq i \leq k$. Associate to $T$ an element $\omega(T) \in A_q(n, r)$, called a (one-column) bideterminant by

$$\omega(T) = \det_q X^{1,2,\ldots,k}_{a_1,a_2,\ldots,a_k}.$$

The following lemma follows from the relations (3).

Lemma 4.1. Let $T$ be a one-column $\Lambda_k$-tableau. Then

1. $\omega(T) = 0$ if $T$ contains repeated entries and
2. if $T$ is column increasing and $T = S \sigma$ then $\omega(T) = (-q)^{\ell(S)} \omega(S)$.

The $\mathbb{C}(q)$-vector space generated by one-column bideterminants $\omega(T)$ given by $\Lambda_k$-tableaux is a $U_q(\mathfrak{gl}_n)$-module, called a fundamental module, with action given by (4); we denote this $U_q(\mathfrak{gl}_n)$-module by $V(\Lambda_k)$. We have the following lemma, which follows readily by use of the relations (3).

Lemma 4.2. Let $T$ be a one-column $\Lambda_k$-tableau with $\omega(T) \neq 0$. 
6 ANNA STOKKE

(1) If $T$ contains an $i + 1$, then $E_i\omega(T) = \omega(S)$ where $S$ is the same as $T$ except that the $i + 1$ has been replaced with an $i$. If $T$ does not contain an $i + 1$, then $E_i\omega(T) = 0$.

(2) If $T$ contains an $i$, then $F_i\omega(T) = \omega(S)$ where $S$ is the same as $T$ except that the $i$ has been replaced by an $i + 1$. If $T$ does not contain an $i$, then $F_i\omega(T) = 0$.

(3) If $T$ contains an $i$, then $K_i\omega(T) = q\omega(T)$ and $K_i\omega(T) = \omega(T)$ otherwise.

Let $\lambda = \sum_{i=1}^n a_iA_i \in \Lambda^+(n, r)$ and let

$$W(\lambda) = V(\Lambda_n)^{\otimes a_n} \otimes V(\Lambda_{n-1})^{\otimes a_{n-1}} \otimes \ldots \otimes V(\Lambda_1)^{\otimes a_1}.$$ 

A basis for $W(\lambda)$ is given by

$$\mathcal{B}_W(\lambda) = \{\omega(T) \mid T \in \text{CT}(\lambda, n)\}.$$ 

Define $w_\lambda \in W(\lambda)$ to be the tensor product of the highest-weight vectors of each $V(\Lambda_k)$. Then $w_\lambda$ has weight $\lambda$ and is the unique highest-weight vector (up to scalars) in $W(\lambda)$. The $U_q(\mathfrak{gl}_n)$-module $V(\lambda) = U_q(\mathfrak{gl}_n)w_\lambda$ is irreducible and every finite dimensional irreducible polynomial $U_q(\mathfrak{gl}_n)$-module is isomorphic to some $V(\lambda)$ where $\lambda \in \Lambda^+(n, r)$. A basis for $V(\lambda)$ is indexed by the elements $T \in \text{SST}(\lambda, n)$ (see, for instance, [9]).

The canonical basis or (lower) global basis for $U_q(\mathfrak{g})^-$, where $\mathfrak{g}$ is a complex simple Lie algebra, was first introduced by Lusztig in [14]. Another proof of the existence of canonical bases was later given by Kashiwara in [12]. The canonical bases induce bases for $V(\lambda)$. For a general introduction to crystal bases, see [8] or [10]. Following [13], we recall the definition of the global crystal basis of a $U_q(\mathfrak{gl}_n)$-module $V(\lambda)$.

Let $\mathcal{A}$ be the subring of $\mathbb{C}(q)$ of rational functions without pole at $q = 0$. Let $L_W(\lambda)$ denote the $\mathcal{A}$-lattice in $W(\lambda)$ spanned by the basis elements in $\mathcal{B}_W(\lambda)$, which is the crystal lattice of $W(\lambda)$. Let $L_V(\lambda) = L_W(\lambda) \cap V(\lambda)$, which is the crystal lattice of $V(\lambda)$.

Define a $\mathbb{C}(q)$-algebra homomorphism on $U_q(\mathfrak{gl}_n)$ that is an involution by

$$(5) \quad E_i = E_i, \quad F_i = F_i, \quad q = q^{-1}, \quad K_j = K_j^{-1}, \quad 1 \leq i < n, \quad 1 \leq j \leq n,$$

and define $w_\lambda = \pi w_\lambda$, where $w = uw_\lambda$ for $u \in U_q(\mathfrak{gl}_n)$.

Let $U_Q^-$ denote the $U_q(\mathfrak{gl}_n)$-subalgebra generated over $\mathbb{Q}[q, q^{-1}]$ by the divided powers $F_i^{(k)} := \frac{F_i^k}{[k]!}$, where $[k]! = [k][k-1]\cdots[1]$ and $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$, and let $V_Q(\lambda) = U_Q^-w_\lambda$. We have the following theorem (see [14] and [12]).

**Theorem 4.3.** There exists a unique $\mathbb{Q}[q, q^{-1}]$-basis $\{G(T) \mid T \in \text{SST}(\lambda, n)\}$ of $V_Q(\lambda)$ with the properties that

1. $G(T) = \omega(T)$ mod $qL_W(\lambda)$,
2. $G(T) = G(T)$.

This basis is called the global crystal basis of $V(\lambda)$.

We now recall the monomial basis for $V(\lambda)$ which was introduced in [13]. Given a semistandard $\lambda$-tableau $T$, let $i$ be the smallest integer such that $i + 1$ appears in
Terminates to give \( T \). Let \( r_1 \) be the number of occurrences of \( i + 1 \) that appear in any row with row number less than \( i + 1 \) and let \( i_1 = i \). Form a new \( \lambda \)-tableau \( T_1 \) by replacing the \( r_1 \) occurrences of \( i + 1 \) by \( i \). Repeat the procedure with \( T_1 \) to give integers \( r_2 \) and \( i_2 \) and a tableau \( T_2 \). After the procedure terminates to give \( T(\lambda) \), we obtain two sequences \((i_1, i_2, \ldots, i_s)\) and \((r_1, r_2, \ldots, r_s)\).

Define \( a(T) = F_{i_1}^{(r_1)} \cdots F_{i_s}^{(r_s)} \in U_q(\mathfrak{g}_n)^- \).

Example 4.4. If \( T = \left( \begin{array}{cccc} 1 & 2 & 2 & 3 & 4 \\ 3 & 4 & \end{array} \right) \) then \( a(T) = F_1^{(2)} F_2^{(2)} F_1 F_3^{(3)} F_2^{(3)} F_1 \).

Given two column increasing \( \lambda \)-tableaux \( S \) and \( T \), let \( S < T \) if \( I_C(S) < I_C(T) \). Lemmas 4.5–4.6 and Theorem 4.7 are proved in [13].

Lemma 4.5. Let \( T \in SST(\lambda, n) \) and suppose that \( a(T)w_\lambda = \sum_{S \in CT(\lambda,n)} \alpha_{ST}(q) \omega(S) \) as a linear combination of basis elements in \( B_W(\lambda) \). Then \( \alpha_{ST}(q) \in \mathbb{N}[q, q^{-1}] \), \( \alpha_{TT} = 1 \) and \( \alpha_{ST}(q) \neq 0 \) only if \( S \geq T \). Furthermore, \( \alpha_{ST}(q) = 0 \) unless \( \omega(S) \) and \( \omega(T) \) have the same weight.

It follows from the above lemma that \( \{a(T)w_\lambda \mid T \in SST(\lambda, n)\} \) is a basis for \( V(\lambda) \). In the lemma and theorem below, let \( \{G(T) \mid T \in SST(\lambda, n)\} \) be the global basis for \( V(\lambda) \).

Lemma 4.6. Let \( T \in SST(\lambda, n) \) and suppose that the expansion of \( G(T) \) in the basis \( \{a(T)w_\lambda \mid T \in SST(\lambda, n)\} \) is \( G(T) = \sum_{S \in SST(\lambda,n)} \beta_{ST}(q)a(S)w_\lambda \). Then

\[
\beta_{TT}(q) = 1, \quad \beta_{ST}(q) = 0 \quad \text{unless} \quad S \geq T.
\]

Theorem 4.7. Let \( T \in SST(\lambda, n) \) and suppose that \( G(T) = \sum_{S \in CT(\lambda,n)} d_{ST}(q) \omega(S) \) as a linear combination of basis elements in \( B_W(\lambda) \). Then

1. \( d_{ST}(q) \in \mathbb{Z}[q] \),
2. \( d_{TT}(q) = 1 \) and \( d_{ST}(0) = 0 \) if \( S \neq T \),
3. \( d_{ST}(q) = 0 \) unless \( \omega(S) \) and \( \omega(T) \) have the same weight and \( S \geq T \).

Using the above, one can obtain the global crystal basis \( \{G(T) \mid T \in SST(\lambda, n)\} \) by a triangular algorithm. Let \( T^{(1)}, T^{(2)}, \ldots, T^{(t)} \) be the tableaux in \( SST(\lambda, n) \) numbered such that \( T(\lambda) = T^{(1)} < T^{(2)} < \cdots < T^{(b)} \). Certainly \( G(T^{(i)}) = a(T^{(i)})w_\lambda \) and, \( G(T^{(i-1)}) = a(T^{(i-1)})w_\lambda - \gamma_i(q)G(T^{(i)}) \), where \( \gamma_i(q) \in \mathbb{Q}[q, q^{-1}] \). Since \( G(T^{(i)}) = G(T^{(i)}) \mod qL_W(\lambda) \), so writing \( a(T^{(i-1)})w_\lambda - \gamma_i(q)G(T^{(i)}) \) as a linear combination of basis elements in \( B_W(\lambda) \) and using these two facts determines \( \gamma_i(q) \).

More generally, if one has written each of \( G(T^{(i+1)}), G(T^{(i+2)}), \ldots, G(T^{(t)}) \) as a linear combination of basis vectors in \( B_W(\lambda) \), then the coefficients in the linear combination \( G(T^{(i)}) = a(T^{(i)})w_\lambda - \gamma_{i+1}(q)G(T^{(i+1)}) - \cdots - \gamma_t(q)G(T^{(t)}) \) are completely determined by the facts that

\[
\gamma_k(q^{-1}) = \gamma_k(q), \quad 1 \leq k \leq t, \quad G(T^{(i)}) \equiv \omega(T^{(i)}) \mod qL_W(\lambda).
\]

For an example, see [13] or Example 7.5.
5. Carter-Lusztig Bases and \( q \)-Schur Algebras

In [18], a quantum version of the Carter-Lusztig basis of the \( q \)-Weyl module, which is isomorphic to \( V(\lambda) \) as a \( U_q(\mathfrak{g}_n) \)-module, is given using elements in \( U_q(\mathfrak{g}_n) \)^+. In [3], it is shown that the elements in the Carter-Lusztig basis can be written in terms of elements in the \( q \)-Schur algebra up to a power of \( q \). Since the \( q \)-Schur algebra version of this basis is easier to work with than the \( U_q(\mathfrak{g}_n) \) version, we use it to prove that this basis is related to the Leclerc-Toffin basis by an upper triangular matrix and provide a method for writing elements in the Leclerc-Toffin basis using elements in the \( q \)-Schur algebra. We then adjust the Leclerc-Toffin algorithm to obtain the global basis for \( V(\lambda) \) in terms of elements in the \( q \)-Schur algebra. We first recall the construction of the quantum Carter-Lusztig basis.

Define \( F_{i,i+1} = F_i \) and for \(|i-j| \geq 1 \) define \( F_{ij}, E_{ij} \in U_q(\mathfrak{g}_n) \) recursively as

\[
F_{ij} = F_{i+1,j}F_i - q^{-1}F_iF_{i+1,j}, \quad E_{ij} = E_iE_{i+1,j} - q^{-1}E_{i+1,j}E_i.
\]

For a semistandard \( \lambda \)-tableau \( T \) with \( k \leq n \) rows, define \( F_T, E_T \in U_q(\mathfrak{g}_n) \) by

\[
F_T = \prod_{1 \leq i < k, i < j \leq n} F_{ij}^{(\gamma_{ij})} F_{12}^{(\gamma_{12})} F_{13}^{(\gamma_{13})} \ldots F_{ik}^{(\gamma_{ik})} F_{23}^{(\gamma_{23})} F_{24}^{(\gamma_{24})} \ldots F_{k-1,k}^{(\gamma_{k-1,k})}, \quad E_T = \prod_{1 \leq i < k, i < j \leq n} E_{ij}^{(\gamma_{ij})} E_{k-1,k}^{(\gamma_{k-1,k})} \ldots E_{23}^{(\gamma_{23})} E_{24}^{(\gamma_{24})} E_{k}^{(\gamma_{k,k})} \ldots E_{k-1,k}^{(\gamma_{k-1,k})},
\]

where \( \gamma_{ij} \) is the number of \( j \)'s in row \( i \) of \( T \), and \( k \) is the number of columns in \( T \).

For \( I = (i_1, i_2, \ldots, i_r) \in I(n, r) \), let \( v_I = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} \in V^{\otimes r} \). Define a bilinear form \( \langle \cdot, \cdot \rangle : V^{\otimes r} \times V^{\otimes r} \to \mathbb{Q}[q, q^{-1}] \) by \( \langle v_I, v_J \rangle = \delta_{I, J} \). The following Lemma reveals the relationship between the two comultiplications \( \Delta \) and \( \Delta_1 \).

**Lemma 5.1.** Let \( u \in U_q(\mathfrak{g}_n) \), \( v, w \in V^{\otimes r} \). Then \( \langle \Delta_1^{-1}(u)v, w \rangle = \langle v, \Delta_1^{-1}(\tau(u))w \rangle \).

**Proof.** It suffices to prove that \( \langle \Delta_1^{-1}(F_i)v_I, v_J \rangle = \langle v_I, \Delta_1^{-1}(E_i)v_J \rangle \), where \( 1 \leq i < n \) and \( I = (i_1, i_2, \ldots, i_r) \), \( J = (j_1, j_2, \ldots, j_r) \in I(n, r) \).

We have

\[
\Delta_1^{-1}(F_i)v_I = v_{i_1} \otimes \cdots \otimes v_{i_{r-1}} \otimes (F_i v_{i_r}) + \cdots + (F_i v_{i_1}) \otimes (K_{i,i+1} v_{i_2}) \otimes \cdots \otimes (K_{i,i+1} v_{i_r})
\]

and

\[
\Delta_1^{-1}(E_i)v_J = v_{j_1} \otimes \cdots \otimes v_{j_{r-1}} \otimes (E_i v_{j_r}) + \cdots + (E_i v_{j_1}) \otimes (K_{i,i+1} v_{j_2}) \otimes \cdots \otimes (K_{i,i+1} v_{j_r}).
\]

Since \( \langle v_I, v_{i_1} \otimes \cdots \otimes (F_i v_{i_k}) \otimes \cdots \otimes (K_{i,i+1} v_{i_{r-1}}) \otimes (K_{i,i+1} v_{i_r}) \rangle \) is the same as \( \langle v_I, v_{j_1} \otimes \cdots \otimes (E_i v_{j_k}) \otimes \cdots \otimes (K_{i,i+1} v_{j_{r-1}}) \otimes (K_{i,i+1} v_{j_r}) \rangle \), for \( 1 \leq k \leq r \), the result follows. \( \square \)

Note that in the proofs below we will simply write \( uv \) instead of \( \Delta_1^{-1}(u)v \) for \( u \in U_q(\mathfrak{g}_n) \) and \( v \in V^{\otimes r} \) but when we are using the action of \( V^{\otimes r} \) given by the comultiplication \( \Delta_1 \), this will always be specified.

Given \( I = (i_1, i_2, \ldots, i_r) \in I(n, r) \), let \( \beta(I) = |\{(a, b) | a < b \text{ and } i_a \neq i_b\}| \). From [18], we have both the following identity and Theorem 5.2:

\[
q^{\beta(I)} \langle \Delta_1^{-1}(u)v_I, v_J \rangle = q^{\beta(I)} \langle v_I, \Delta_1^{-1}(\tau(u))v_J \rangle.
\]

**Theorem 5.2.** The set \( \{F_T w_\lambda | T \in SST(\lambda, n)\} \) is a basis for \( V(\lambda) \).
Proof. In [18], it is proved that \( \{ \Delta_{I}^{-1}(F_{T})z_{\lambda} \mid T \in \text{SST}(\lambda, n) \} \) is a basis for the \( q \)-Weyl module, \( \Delta_{q}(\lambda) \), which is the \( U_{q}(\mathfrak{gl}_{n}) \)-submodule of \( V^{\otimes r} \), generated by the highest-weight vector \( z_{\lambda} = \sum_{\sigma \in C(\lambda)} (-q)^{-L(\sigma)}v_{I(\lambda)\sigma} \in V^{\otimes r} \). For a given \( T \in \text{SST}(\lambda, n) \) with \( I_{R}(T) = J \), write \( F_{T}z_{\lambda} = \sum_{K \in I(n, r)} a_{K}v_{K} \), as a linear combination of basis elements in \( V^ {\otimes r} \). Since each \( K \) in the sum has \( K = J\sigma \) for some \( \sigma \in S_{r} \), \( \beta(K) = \beta(J) \). Furthermore, \( a_{K} = \langle F_{T}z_{\lambda}, v_{K} \rangle = \langle z_{\lambda}, \Delta_{I}^{-1}(F_{T})z_{\lambda} \rangle \).

Since \( \beta(I_{C}(\lambda)\sigma) = \beta(I(\lambda)) \) for \( \sigma \in C(\lambda) \), for each \( K \) we have
\[
\langle z_{\lambda}, \Delta_{I}^{-1}(F_{T})v_{K} \rangle = q^{\beta(K)-\beta(I(\lambda))} \langle \Delta_{I}^{-1}(F_{T})z_{\lambda}, v_{K} \rangle = q^{\beta(J)-\beta(I(\lambda))} \langle \Delta_{I}^{-1}(F_{T})z_{\lambda}, v_{K} \rangle.
\]
It follows that \( F_{T}z_{\lambda} = q^{\beta(J)-\beta(I(\lambda))} \Delta_{I}^{-1}(F_{T})z_{\lambda} \) so that \( \{ F_{T}z_{\lambda} \mid T \in \text{SST}(\lambda, n) \} \) is a basis for \( \Delta_{q}(\lambda) \). Since the highest weight module \( \Delta_{q}(\lambda) \) is isomorphic to \( V(\lambda) \), the theorem now follows. \( \square \)

Write \( T_{I} \) for the tableau \( T \in CT(\lambda, n) \) with column sequence \( I \). Then \( W(\lambda) \) is an \( S_{q}(n, r) \)-module, with action \( \xi\omega(T_{I}) = \sum_{\lambda \in I(n, r)} \xi(x_{A, I})\omega(T_{A}) \).

Given \( v_{I} \in V^{\otimes r} \) and \( u \in U_{q}(\mathfrak{gl}_{n}) \), define \( \theta : U_{q}(\mathfrak{gl}_{n}) \to S_{q}(n, r) \) by \( \theta(u)(x_{I, J}) = \langle uv_{I}, v_{J} \rangle \). The following lemma is proved in [11, Lemma 5.1, 5.2].

**Lemma 5.3.** Let \( \theta : U_{q}(\mathfrak{gl}_{n}) \to S_{q}(n, r) \) be as defined above, let \( u, w \in U_{q}(\mathfrak{gl}_{n}) \) and \( T \in CT(\lambda, n) \). Then

1. \( \theta(uw) = \theta(u)\theta(w) \) and
2. \( \theta(u)\omega(T) = uw(T) \).

Define \( \left( K_{i} \right)^{t} = \prod_{i=1}^{t} \frac{q^{s+1}K_{i} - q^{s-1}K_{i-1}}{q^{s} - q^{-s}} \in U_{q}(\mathfrak{gl}_{n}) \), for \( 1 \leq i, t \leq n \). Suppose that \( \lambda = (\lambda_{1}, \ldots, \lambda_{k}) \), let \( u_{i} = \left( K_{i} \right)^{t} \) and define \( u^{\lambda} = \prod_{i=1}^{k} u_{i} \in U_{q}(\mathfrak{gl}_{n}) \).

**Lemma 5.4.** For each \( \lambda \in \Lambda^{+}(n, r) \), we have \( \theta(u^{\lambda}) = \xi_{I(\lambda), I(\lambda)} \).

**Proof.** We will prove that \( \theta(u^{\lambda}) = \xi_{I(\lambda), I(\lambda)} \) by showing that \( u^{\lambda}v_{I(\lambda)\sigma} = v_{I(\lambda)\sigma} \) for \( \sigma \in S_{r} \) and that \( u^{\lambda}v_{J} = 0 \) for \( J \in I(n, r) \) when \( J \neq I(\lambda)\sigma \) for any \( \sigma \in S_{r} \). Since
\[
\prod_{i=1}^{k} \frac{q^{s+1}K_{i} - q^{s-1}K_{i-1}}{q^{s} - q^{-s}} = 1,
\]
we have
\[
u^{\lambda}v_{I(\lambda)\sigma} = \prod_{i=1}^{k} \frac{q^{s+1}K_{i} - q^{s-1}K_{i-1}}{q^{s} - q^{-s}}v_{I(\lambda)\sigma} = \prod_{i=1}^{k} \frac{q^{s+1}q^{\lambda_{i}} - q^{-1}q^{-\lambda_{i}}}{q^{s} - q^{-s}}v_{I(\lambda)\sigma} = v_{I(\lambda)\sigma}.
\]
Consider \( J \in I(n, r) \), with \( J \neq I(\lambda)\sigma \) for any \( \sigma \in S_{r} \). There must be some \( m \) with \( 1 \leq m \leq k \) that appears \( a_{m} \) times in the \( r \)-tuple \( J \) with \( a_{m} < \lambda_{m} \); let
Theorem 5.5. Let $m$ be maximal with this property. Then $u^λv_J = \prod_{i=1}^k u_i v_J = \prod_{i=1}^m u_i(α(q)v_J)$, where $α(q) ∈ \mathbb{Q}[q, q^{-1}]$ and

$$u_m v_J = \left( \frac{K_m}{λ_m} \right) v_J = \prod_{s=1}^{m} \frac{q^{-s+1}K_m - q^{-s-1}K_m^{-1}}{q^s - q^{-s}} \frac{q^{-α_m}K_m - q^α_m K_m^{-1}}{q^{α_m+1} - q^{-(α_m+1)}} β(q)v_J,$$

where $β(q) ∈ \mathbb{Q}[q, q^{-1}]$. But $(q^{-α_m}K_m - q^α_m K_m^{-1})v_J = (q^{-α_m}q^α_m - q^α_m q^{-α_m})v_J = 0$, so that $u^λv_J = 0$. □

Let $T ∈ SST(λ, n)$. Denote the entry in the $i$-th row and $j$-th column of $T$ by $T_{ij}$ and define $s(T) = |\{(i, j, a, b) | i > j, a < b, T_{ia} = T_{jb}\}|$. By definition, $s(T)$ counts the number of pairs $(ia, jb)$ for which $T_{ia} = T_{jb}$ and $T_{ia}$ sits in a row below $T_{jb}$ and in a column to the left of $T_{jb}$. Define $r(T) = |\{(i, a, b) | a < b ≤ λ_i, T_{ia} ≠ T_{ib}\}|$.

The following theorem is an adjusted version of [3, Theorems 18, 19].

**Theorem 5.6.** Let $T ∈ SST(λ, n)$ with $J = I_R(T)$. Then

1. $θ(F_T)ξ_{I(λ)} = q^{-s(T)}ξ_{I(λ)}$
2. $ξ_{I(λ)}θ(E_T) = q^{-r(T)}ξ_{I(λ)}$.

**Proof.** In [3] it was proved that $\langle v_{I(λ)}, ξ_{I(λ)} Δ_1^{-1}(E_T)v_K \rangle = 0$ unless $K = J$ and that $\langle v_{I(λ)}, Δ_1^{-1}(E_T)ξ_{I(λ)}v_J \rangle = q^{-s(T)}$. By Lemma 3.2, $θ(F_T)ξ_{I(λ)} = θ(F_T u^λ) = \sum_{Q ∈ R(λ, n)} a_Q ξ_{Q,I(λ)}$. Since

$$a_Q = \langle F_T u^λ v_{I(λ)}, v_Q \rangle = \langle v_{I(λ)}, Δ_1^{-1}(u^λ E_T)v_Q \rangle = \langle v_{I(λ)}, ξ_{I(λ)} Δ_1^{-1}(E_T)v_Q \rangle,$$

we have $a_Q = 0$ unless $Q = J$ and $a_J = q^{-s(T)}$.

It was also proved in [3] that $\langle v_{I(λ)}, ξ_{I(λ)} Δ_1^{-1}(F_T)v_K \rangle = 0$ unless $Q = J$ and that $\langle v_{I(λ)}, Δ_1^{-1}(F_T)v_J \rangle = q^{-r(T)}$ from which the second statement follows similarly. □

Let $S = \{(λ, I, J) | λ ∈ \Lambda^+(n, r), I = I_R(T), J = I_R(S) \}$ for $S, T ∈ SST(λ, n)$. The main result in [3] gives a codeterminant basis for $S_q(n, r)$.

**Theorem 5.6.** The set $\{ξ_{A,I(λ)} | (λ, A, B) ∈ S \}$ is a basis for $S_q(n, r)$.

The following follows immediately from Theorems 5.5 and 5.6 and Lemma 5.4.

**Theorem 5.7.** The map $θ : U_q(\mathfrak{gl}_n) → S_q(n, r)$ is surjective.

**Remark 2.** In [1], another version of the $q$-Schur algebra is defined using structure constants arising from flags in vector spaces over a field of $q$ elements, and a surjective map from $U_q(\mathfrak{gl}_n)$ to the $q$-Schur algebra is also given in that setting.

**Corollary 5.8.** Let $λ ∈ \Lambda^+(n, r)$. Then $V(λ) = \{ξw_λ | ξ ∈ S_q(n, r)\}$ and the set $\{ξ_{I,I(λ)} w_{I(λ)} | J = I_R(T), T ∈ SST(λ, n)\}$ is a basis for $V(λ)$.

**Proof.** The first part of the statement follows from Lemma 5.3 and Theorem 5.7 and the second part from Theorems 5.2 and 5.5 and Lemma 5.3. □
We can reformulate Theorem 4.3 in terms of the $q$-Schur algebra by first defining a map $- : S_q(n, r) \to S_q(n, r)$ by
\[
\langle u \rangle = \theta(\pi), \quad \text{where } \eta = \theta(u) \in S_q(n, r), \ u \in U_q(\mathfrak{g}l_n).
\]
Then a map $V(\lambda) \to V(\lambda)$ is given by $\xi w_\lambda = \xi w_\lambda$. Note that if $\xi = \theta(u)$, then
\[
\xi w_\lambda = \theta(u) w_\lambda = \theta(\pi) w_\lambda = \eta w_\lambda \equiv \eta w_\lambda \pmod{\lambda}.
\]

For the next example, consider that if $u \in U_q(\mathfrak{g}l_n)$ and the expansion of $\theta(u)|_{I(\lambda), I(\lambda)}$ on basis elements in $S_q(n, r)$ is given by $\theta(u)|_{I(\lambda), I(\lambda)} = \sum_{Q \in R(\lambda, n)} a_Q|_{Q, I(\lambda)}$, then
\[
(a_Q = \theta(u)|_{I(\lambda), I(\lambda)}(x_Q, I(\lambda)) = \theta(u)(x_Q, I(\lambda)) = \langle uv, I(\lambda), v_Q \rangle).
\]

Also note that $u^\lambda = w^\lambda$ so that $\xi|_{I(\lambda), I(\lambda)} = \xi|_{I(\lambda), I(\lambda)}$.

**Example 5.9.** Let $\lambda = (2, 1)$ and let $T_1 = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ and $T_2 = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$.

Then $\xi|_{(1, 2, 3), (1, 1, 2)} = \theta(F_1|_{T_1}, u^\lambda) = \theta(F_1|_{T_2}, u^\lambda) = \xi|_{(1, 2, 3), (1, 1, 2)}$, and $\xi|_{(1, 1, 3), (1, 1, 2)} = \theta(F_2|_{F_1, u^\lambda} - qF_1|_{F_2, u^\lambda})|_{I(\lambda), I(\lambda)} = \theta(F_2|_{F_1, u^\lambda} - qF_1|_{F_2, u^\lambda})|_{I(\lambda), I(\lambda)} - q|_{(1, 2, 3), (1, 1, 2)}$.

We have $\theta(F_2|_{F_1, u^\lambda})|_{I(\lambda), I(\lambda)} = \sum_{Q \in R(\lambda, n)} a_Q|_{Q, I(\lambda)}$, where $a_Q = \langle F_2|_{F_1, u^\lambda}, v_Q \rangle$. Calculating $F_2|_{F_1, u^\lambda}$ and extracting coefficients of basis elements $v_Q$, where $Q$ gives the row sequence of a row increasing tableau, yields $\theta(F_2|_{F_1, u^\lambda})|_{I(\lambda), I(\lambda)} = \xi|_{(1, 1, 3), (1, 1, 2)} + q^{-1} \xi|_{(1, 2, 3), (1, 1, 2)}$. Thus
\[
\xi|_{(1, 1, 3), (1, 1, 2)} = \xi|_{(1, 1, 3), (1, 1, 2)} - (q - q^{-1}) \xi|_{(1, 2, 3), (1, 1, 2)}.
\]

The following theorem is a version of Theorem 4.3 in terms of elements from the $q$-Schur algebra.

**Theorem 5.10.** Suppose that an element $\xi_T \in S_q(n, r)$ is defined for each $T \in SST(\lambda, n)$. The set $\{\xi_T w_\lambda \mid T \in SST(\lambda, n)\}$ is the global crystal basis for $V(\lambda)$ if the following properties are satisfied for each $T \in SST(\lambda, n)$:

1. As a linear combination of basis elements in $\mathcal{B}_W(\lambda)$, we have $\xi_T w_\lambda = \sum_{S \in CT(\lambda, n)} \alpha_S \omega(S)$, where $\alpha_S \in \mathbb{Z}[q]$,
2. $\xi_T w_\lambda \equiv \omega(T) \pmod{qL_W(\lambda)}$,
3. $\xi_T w_\lambda = \xi_T w_\lambda$.

**Proof.** Suppose that, for each $T \in SST(\lambda, n)$, we have $\xi_T = \theta(u_T)$ where $u_T \in U_q(\mathfrak{g}l_n)$. By Lemma 5.3, $\xi_T w_\lambda = \theta(u_T) w_\lambda = u_T w_\lambda$. We have $u_T w_\lambda = \xi_T w_\lambda = \xi_T w_\lambda = \xi_T w_\lambda = u_T w_\lambda \equiv \omega(T) \pmod{qL_W(\lambda)}$. Thus $\{u_T w_\lambda \mid T \in SST(\lambda, n)\} = \{\xi_T w_\lambda \mid T \in SST(\lambda, n)\}$ is the global crystal basis for $V(\lambda)$. \qed
Example 5.11. Referring to Example 5.9, if \( \lambda = (2, 1) \), then the set
\[
\{ \xi_{(1,2,3),(1,1,2)} w_\lambda, \xi_{(1,3,2),(1,1,2)} + q^{-1} \xi_{(1,2,3),(1,1,2)} w_\lambda \}
\]
is the portion of the global crystal basis corresponding to the weight space \( V(\lambda)^\chi \), where \( \chi = (1, 1, 1) \).

6. Relationsips between bases

We shall say that a tableau \( T \) is diagonally related to a \( \lambda \)-tableau \( S \), \( T \triangleright_D S \), if \( S \) can be obtained from \( T \) by exchanging an entry \( a \) in \( T \) with an entry \( b > a \) where \( a \) sits in a row below \( b \) and in a column left of \( b \). Define \( \triangleright_D \) to be the partial order defined by extending \( \triangleright_d \) reflexively and transitively.

Example 6.1. We have

\[
\begin{array}{ccc}
1 & 3 & \triangleright_d \ 1 & 2 & \triangleright_d \ 2 & 4 & \triangleright_d \\
2 & 4 & \triangleright_d 3 & 4 & \triangleright_d \\
3 & \quad & 3 & \quad & 4 \\
\end{array}
\]

Recall that if \( T \) has row sequence \( Q \in I(n, r) \), we denote the column sequence of \( T \) by \( Q' \).

Lemma 6.2. Let \( \lambda \in \Lambda^+(n, r) \) and suppose that \( x_{M,I_C(\lambda)} = \sum_{K \in R(\lambda,n)} a_k x_{K,I(\lambda)} \) as a linear combination of basis elements. Then, if \( a_K \neq 0 \), we have \( T_{K^*} \sim_R W \triangleright_D T_M \) for some tableau \( W \).

Proof. We will use a specific recipe for rewriting \( x_{M,I_C(\lambda)} \) as a linear combination of basis elements. Starting with \( i = 1 \), and the left-most \( x_{m_i} \) in \( x_{M,I_C(\lambda)} \), use the relations (3) to move \( x_{m_i} \) left of all \( x_{s_j} \) where \( x_{s_j} \) sits left of \( x_{m_i} \) and \( j > i \). Repeat this procedure for \( i = 2, \ldots, \mu_1 \), where \( \mu = (\mu_1, \ldots, \mu_{\lambda_1}) \) is the conjugate partition, and then for each of the resulting summands to get

\[
(8) \quad x_{M,I_C(\lambda)} = \sum_B c_B x_{B,I(\lambda)}.
\]

Now rewrite each \( x_{B,I(\lambda)} \) in the sum using the second of the relations (3) to get

\[
\sum_K a_K x_{K,I(\lambda)} \text{ where each } (K,I(\lambda)) \text{ in the sum satisfies } (K,I(\lambda))_0 = (K,I(\lambda)).
\]

If \( a_K \neq 0 \), then one possibility is that \( (M,I_C(\lambda))_0 = (K,I(\lambda)) \), in which case \( T_{K^*} \sim_R T_M \), and in this case, \( W = T_{K^*} \). Otherwise, the fourth property of relations (3) was used at least once in the above procedure which resulted in \( x_{K,I(\lambda)} \) in the sum. There is then an \( x_{B,I(\lambda)} \) in the first sum (8) with \( (B,I(\lambda))_0 = (K,I(\lambda)) \) (in other words, \( T_{K^*} \sim_R T_{B'} \)) and the fourth relation was used at least once in rewriting \( x_{M,I_C(\lambda)} \) to get \( x_{B,I(\lambda)} \) in the sum (8). We will show that \( T_{B'} \triangleright_D T_M \).

Since the fourth relation was applied to \( x_{M,I_C(\lambda)} \), we have

\[
\begin{align*}
x_{M,I_C(\lambda)} &= \cdots x_{m_1,j_1} \cdots x_{m_2,j_2} \cdots \\
&= \alpha(q) x_{m_2,j_1} \cdots x_{m_1,j_2} \cdots + \text{ other terms} \\
&= \alpha(q) x_{M,I_C(\lambda)} + \text{ other terms},
\end{align*}
\]

where \( m_1 > m_2 \) and \( j_1 > j_2 \). Since \( j_1 > j_2 \), in the tableau \( T_M \) we have \( m_1 > m_2 \) and \( m_1 \) sits southwest of \( m_2 \). It follows that \( T_{M_1} \triangleright_d T_M \). Either \( (B,I_C(\lambda))_0 = (M,I_C(\lambda))_0 \) or the fourth relation can be applied again to \( x_{M_1,I_C(\lambda)} \) to get \( T_{M_2} \triangleright_d
Since all columns prior to this column contain the same entries in both $T$ and $T'$, consider the left-most column where that diagonal exchanges, which again increase entries. Thus the column sequence of $T$ is not column increasing, in which case $\omega(T_A) = \pm \omega(T)$, where $T_A \sim C T$ and $T$ is column increasing, so that $\omega(T) \in B_W(\lambda)$. 

**Lemma 6.4.** Suppose that $T \in SST(\lambda, n)$ and that $T \sim_R W \triangleright_D S$ for $\lambda$-tableaux $W$ and $S$. Then $T \preceq_C S$ and, if $U$ is equal to the $\lambda$-tableau obtained by rewriting the columns of $S$ in increasing order, then $T \preceq_C U$.

**Proof.** If $T \sim_R W$, then $T \preceq_C W$. Furthermore, if $W \triangleright_d W_1 \triangleright_d \cdots \triangleright_d W_k \triangleright_d S$, where $W \neq S$, an inductive argument shows that $W \preceq_C S$ so that $T \preceq_C S$. To see that $T \preceq_C U$, where $U$ comes from $S$ by rewriting its columns to be increasing, consider the left-most column where $T$ and $U$ differ. Since all columns prior to this column contain the same entries in both $T$ and $U$, the smallest entry in this column of $U$ that is different from one in $T$ must have arisen through a row exchange with an entry larger than one in $T$, possibly combined with a number of diagonal exchanges, which again increase entries. Thus the column sequence of $T$ associated to this column is less than that of $U$ and so $T \preceq_C U$. 

**Corollary 6.5.** Let $T \in RT(\lambda, n)$ and let $Q$ be the row sequence of $T$. Then $\xi_{Q,I}(\lambda)w_\lambda = \sum_{T_A \in CT(\lambda, n)} b_A \omega(T_A)$, where for each $T_A$ in the sum, $T_{Q'} \preceq_C T_A$ and $b_Q' = q^{(Q', I_C(\lambda))}$.

**Proof.** We have $\xi_{Q,I}(\lambda)w_\lambda = \sum_{A \in J(n,r)} \xi_{Q,I}(x_{A,I_C(\lambda)})\omega(T_A)$ and $\xi_{Q,I}(\lambda)(x_{A,I_C(\lambda)})$ contributes to the coefficient $b_{Q'}$ of $\omega(T_Q)$ if and only if $T_A = T_{Q'}\sigma$ for some $\sigma \in C(\lambda)$. However, using Lemma 6.2, $\xi_{Q,I}(\lambda)(x_{Q',I_C(\lambda)}) = 0$ for $\sigma \in C(\lambda)$ unless $\sigma$ is the identity permutation. Thus we have

$$b_{Q'} = \xi_{Q,I}(\lambda)(x_{Q',I_C(\lambda)}) = \xi_{Q,I}(\lambda)(q^{(Q', I_C(\lambda))})x_{Q,I(\lambda)} + \sum_{(S,T)} x_{S,T},$$

where the pairs $(S, T)$ in the sum satisfy $(S, T)_0 = (S, T)$ and $(S, T) > (Q', I_C(\lambda))$ by Lemma 3.1. It follows that $b_{Q'} = q^{(Q', I_C(\lambda))} \neq 0$. 

Consequently, $T_{K'} \sim_R T_{B'} \triangleright_D T_M$. 

**Lemma 6.3.** Suppose that $S \in RT(\lambda, n)$ and let $Q$ denote the row sequence of $S$. Then, as a $\mathbb{Q}[q, q^{-1}]$-linear combination of basis elements in $B_W(\lambda)$, we have

$$\xi_{Q,I}(\lambda)w_\lambda = \sum_{T \in CT(\lambda, n)} b_T \omega(T),$$

where if $b_T \neq 0$, then $S \sim_R W_1 \triangleright_D W_2 \sim C T$, for some $\lambda$-tableaux $W_1$ and $W_2$.

**Proof.** We have $\xi_{Q,I}(\lambda)w_\lambda = \sum_{A \in J(n,r)} \xi_{Q,I}(x_{A,I_C(\lambda)})\omega(T_A)$. By Lemma 6.2, for each $A$ in the sum we have $x_{A,I_C(\lambda)} = \sum_{(K,I) \in \mathcal{J}} c^A_K x_{K,I(\lambda)}$, where $c^A_K = 0$ unless $T_{K'} \sim_R W_1 \triangleright_D T_A$. Since $(K,I(\lambda)), (Q,I(\lambda)) \in \mathcal{J}$, $\xi_{Q,I}(\lambda)(x_{(K,I(\lambda)}) = 0$ unless $K = Q$. Thus $\xi_{Q,I}(\lambda)w_\lambda = \sum_{A \in J(n,r)} c^A_Q \omega(T_A)$, where for each $A$ in the sum, $S \sim_R W_1 \triangleright_D T_A$. It may be that $T_A$ is not column increasing, in which case $\omega(T_A) = \pm \omega(T)$, where $T_A \sim_C T$ and $T$ is column increasing, so that $\omega(T) \in B_W(\lambda)$. 

However, using Lemma 6.2, $\xi_{Q,I}(\lambda)(x_{Q',I_C(\lambda)}) = 0$ for $\sigma \in C(\lambda)$ unless $\sigma$ is the identity permutation. Thus

$$b_{Q'} = \xi_{Q,I}(\lambda)(x_{Q',I_C(\lambda)}) = \xi_{Q,I}(\lambda)(q^{(Q', I_C(\lambda))})x_{Q,I(\lambda)} + \sum_{(S,T)} x_{S,T},$$

where the pairs $(S, T)$ in the sum satisfy $(S, T)_0 = (S, T)$ and $(S, T) > (Q', I_C(\lambda))$ by Lemma 3.1. It follows that $b_{Q'} = q^{(Q', I_C(\lambda))} \neq 0$. 

Let $SST(\lambda,n) = \{Q \in I(n,r) \mid Q = I_R(T) \text{ for some } T \in SST(\lambda,n)\}$. An immediate consequence of the following theorem is that the global crystal basis and Carter-Lusztig basis for $V(\lambda)$ are related by an upper triangular invertible matrix.

**Theorem 6.6.** Let $T$ be a semistandard $\lambda$-tableau with row sequence $J$ and suppose that $a(T)w_\lambda = \sum_{Q \in SST(\lambda,n)} a_Q \xi_{Q,I(\lambda)} w_\lambda$ is the expansion of $a(T)w_\lambda$ in the basis $\{\xi_{Q,I(\lambda)} w_\lambda \mid Q \in SST(\lambda,n)\}$. Then

1. $a_J = q^{-s(T)}$, and
2. if $a_Q \neq 0$, then $\omega(T_{Q^t})$ and $\omega(T)$ have the same weight and $T \leq C T_{Q^t}$.

**Proof.** The fact that each $Q$ with $a_Q \neq 0$ corresponds to $\omega(T_{Q^t})$ with the same weight as $\omega(T)$ follows from Lemma 4.5 combined with Lemma 6.3. Suppose that some $Q$ in the sum has $T_{Q^t} < T$ and choose $K$ so that $K^t$ is minimal with this property. By Corollary 6.5, when each $\xi_{Q,I(\lambda)} w_\lambda$ is written as a $\mathbb{Q}[q,q^{-1}]$-linear combination of basis elements in $B_W(\lambda)$, $\omega(T_{K^t})$ only appears in $\xi_{K,I(\lambda)} w_\lambda$, and it appears with non-zero coefficient and so appears with non-zero coefficient in the sum $a(T)w_\lambda$, which is not possible by Lemma 4.5.

Thus $a(T)w_\lambda = \sum_{Q \in SST(\lambda,n)} a_Q \xi_{Q,I(\lambda)} w_\lambda = a_J \xi_{J,I(\lambda)} w_\lambda \sum_{Q \in SST(\lambda,n)} a_Q \omega(T_{Q^t})$, where each $Q$ in the sum has $T_{Q^t} > T$. But $a_J \xi_{J,I(\lambda)} w_\lambda = q^{e(J^t,I_C(\lambda))}a_J \omega(T) + \sum_B a_B \omega(T_B)$, where each $\omega(T_B) \in B_W(\lambda)$ with $T_B > T$. Furthermore, $a(T)w_\lambda = \omega(T) + \sum_L c_L \omega(T_L)$ where $T_L > T$. It follows that $a_J = q^{-e(J^t,I_C(\lambda))}$.

Write $J^t = (j_1, j_2, \ldots, j_r)$ and $I_C(\lambda) = (i_1, i_2, \ldots, i_r)$. If $i_a = i_b$, then $j_a$ and $j_b$ belong to the same row and, since $T$ is semistandard, $j_a < j_b$. It follows that $e(J^t, I_C(\lambda)) = \{(a,b) \mid a < b, j_a = j_b, i_a > i_b\}$.

If $j_a$ belongs to column $k$ of $T$ and $j_b$ belongs to column $\ell$, then $j_a = T_{i_a k}$ and $j_b = T_{i_b \ell}$ and, since $T$ is semistandard, $\ell < k$ whenever $j_a = j_b$ and $a < b$. Thus $e(J^t, I_C(\lambda)) = \{(k, \ell, i_a, i_b) \mid \ell < k, T_{i_a k} = T_{i_b \ell}, i_a > i_b\}$.

7. **AN ALGORITHM FOR WRITING THE GLOBAL CRYSTAL BASIS IN TERMS OF ELEMENTS FROM THE $q$-SCHUR ALGEBRA**

The algorithm from [13] allows us to write each element of the global crystal basis vectors from $V(\lambda)$ in terms of elements $a(T)w_\lambda$ from the Leclerc-Toffin basis. The map $\theta : U_q(\mathfrak{gl}_n) \rightarrow S_q(n,r)$ can then be exploited to write each $a(T)w_\lambda$ in terms of elements from the $q$-Schur algebra. We first establish two lemmas which shorten computation time.

If $a(T) = F^{(r)}_1 \cdots F^{(r)}_n \in U_q(\mathfrak{gl}_n)^-$, define $b(T) = \tau(a(T)) = E^{(r)}_1 \cdots E^{(r)}_n \in U_q(\mathfrak{gl}_n)^+$. Since it is often easier to find $\langle v_{I(\lambda)}, b(T)v_Q \rangle$ than $\langle a(T)v_{I(\lambda)}, v_Q \rangle$, the following lemma is quite useful.

**Lemma 7.1.** Let $T \in SST(\lambda,n)$, and let $Q$ denote the row sequence of $T$. Then

$$\langle a(T)v_{I(\lambda)}, v_Q \rangle = q^{r(T)-s(T)}\langle v_{I(\lambda)}, b(T)v_Q \rangle.$$
Proof. Using Lemma 5.1 we have \( \langle a(T)v_{I(\lambda)}, v_Q \rangle = \langle v_{I(\lambda)}, \Delta_1^{-1}(b(T))v_Q \rangle \). By (1),
\[
\langle v_{I(\lambda)}, \Delta_1^{-1}(b(T))v_Q \rangle = q^{\beta(Q) - \beta(I(\lambda))}\langle \Delta_1^{-1}(a(T))v_{I(\lambda)}, v_Q \rangle = q^{\beta(Q) - \beta(I(\lambda))}\langle v_{I(\lambda)}, b(T)v_Q \rangle.
\]

Now, \( \beta(Q) \) counts the number of pairs \( (a, b) \) in \( T \) where \( a \) and \( b \) belong to the same row but \( a < b \) plus the pairs where \( a \neq b \) and \( b \) belongs to a row below \( a \). Furthermore, \( \beta(I(\lambda)) \) counts the pairs \( (a, b) \) in \( T \) where \( b \) sits in a row below \( a \). Thus, \( \beta(Q) - \beta(I(\lambda)) = r(T) - s(T) \). \( \Box \)

The following lemma allows us to classify the \( Q \in \mathcal{R}(\lambda, n) \) that yield a non-zero coefficient \( a_Q \) in the linear combination \( \theta(a(T))\xi_{I(\lambda),I(\lambda)} = \sum_{Q \in \mathcal{R}(\lambda, n)} a_Q\xi_{Q,I(\lambda)} \). We first give a simple example to illustrate the result.

**Example 7.2.** Let \( \lambda = (2, 1) \) and consider the \( \lambda \)-tableau \( T = \begin{array}{c}
1 \\
2 \\
3 \\
\end{array} \). Then
\[
a(T)v_1 \otimes v_1 \otimes v_2 = v_1 \otimes v_3 \otimes v_2 + q^{-1}v_1 \otimes v_2 \otimes v_3 + qv_3 \otimes v_1 \otimes v_2 + v_2 \otimes v_1 \otimes v_3.
\]
Consider the tableaux \( T_{M^t} \) arising from the 3-tuples \( M \) that appear in the linear combination \( a(T)v_{I(\lambda)} = \sum_M a_Mv_M \). We have
\[
\begin{array}{c}
1 & 2 & 3 \\
2 \\
\end{array} \sim_R \begin{array}{c}
3 & 1 & 2 \\
2 \\
\end{array} \quad \text{and} \quad \begin{array}{c}
1 & 3 & 2 \\
2 \\
\end{array} \triangleright_D \begin{array}{c}
1 & 2 & 3 \\
3 \\
\end{array} \sim_R \begin{array}{c}
2 & 1 & 3 \\
3 \\
\end{array}.
\]

**Lemma 7.3.** Suppose that \( T \) is a semistandard \( \lambda \)-tableau. If \( \langle a(T)v_{I(\lambda)}, v_K \rangle \neq 0 \), then \( T \triangleright_D W \sim_R T_{K^t} \), for some \( \lambda \)-tableau \( W \).

**Proof.** We will show that \( a(T)v_{I(\lambda)} = \sum_K a_Kv_K \) where, for each \( a_K \neq 0 \), we have \( T \triangleright_D W \sim_R T_{K^t} \) for some \( \lambda \)-tableau \( W \). To make the connection with Young tableau more readily apparent, we will associate \( v_M \in V^{\otimes r} \) with the tableau \( T_{M^t} \) (not to be confused with \( \omega(T_{M^t}) \) \( \in W(\lambda) \) which would be zero if \( T_{M^t} \) contained two equal column entries, while the corresponding \( v_M \) would not be zero). Instead of writing \( a(T)v_{I(\lambda)} \), for instance, we will write \( a(T)T(\lambda) \) and keep track of the effect of applying the \( F_i \)'s in this way. We write \( F_iT_{M^t} = \sum_B a_Bv_{B^t} \) when \( F_i v_M = \sum_B a_Bv_B \).

The proof is by induction on the number of entries \( \ell \) in \( T \), \( 1 \leq \ell \leq n \), that belong to a row \( r \) with \( \ell \neq r \). Suppose first that there is one such \( \ell \) and let \( r \) be the highest row in \( T \) in which there is an \( \ell \) with \( r < \ell \). Then all \( \ell \)'s in \( T \) appear below row \( r - 1 \) and above row \( \ell + 1 \) and \( a(T)T(\lambda) = F_{\ell - 1, \ell}F_{\ell - 2, \ell - 1} \cdots F_{r,r+1}T(\lambda) \). Suppose that \( T_0 \) is the tableau that comes from \( T \) by changing all \( \ell \)'s above row \( \ell \) to \( \ell - 1 \), \( T_1 \) is the tableau that comes from \( T_0 \) by changing all \( \ell - 1 \)'s above row \( \ell - 1 \) in \( T_0 \) to \( \ell - 2 \), \( \ldots \), \( T_j \) comes from changing all \( r + 2 \)'s above row \( r + 2 \) of \( T_{j-1} \) to \( r + 1 \)'s (in other words, \( T_j \) is the same as \( T(\lambda) \) except that the \( k_j \) rightmost \( r \)'s in row \( r \) have been changed to \( r + 1 \)'s).

Then \( F_{r,r+1}T(\lambda) = T_j + \sum_\alpha T_\alpha \) where the sum \( \sum_\alpha T_\alpha \) runs over the non-semistandard \( T_\alpha \) that come from \( T(\lambda) \) by replacing \( k_j \) entries in row \( r \) with \( r + 1 \); in particular, \( T_\alpha \sim_R T_j \) for each \( \alpha \). Below, we will use the fact that, if \( F_iS = \sum_k a_kT_k \),
for a tableau \( S \), and \( S \sim_R W \), then whenever \( a_m \neq 0 \) in the sum \( F_i W = \sum_m a_m T_m \), we have \( T_m \sim_R T_k \) for some \( a_k \neq 0 \).

Applying \( F^{(k_j-1)}_{r+1,r+2} \) to \( T_j \) yields a sum of tableaux that come from replacing \( k_{j-1} \) entries equal to \( r+1 \) in \( T_j \) with \( r+2 \). If we change the \( k_j \) rightmost \( r+1 \)'s in row \( r \) of \( T_j \) to \( r+2 \)'s and the rightmost \( k_{j-1} - k_j \) entries equal to \( r+1 \) in row \( r+1 \) of \( T_j \), to \( r+2 \)'s, we obtain \( T_{j-1} \). The other tableaux in the sum \( F^{(k_j-1)}_{r+1,r+2} T_j \) are either row equivalent to \( T_j \) or come from changing \( t < k_j \) entries equal to \( r+1 \) in row \( r \) to \( r+2 \) and \( k_{j-1} - t \) entries in row \( r+1 \) to \( r+2 \). If such a tableau \( T_\beta \) is weakly row increasing, then \( T_{j-1} \triangleright_D T_\beta \) by interchanging a series of \( r+1 \)'s in row \( r+1 \) with \( r+2 \)'s in row \( r \). If not, \( T_\beta \) will be row equivalent to such a tableau.

None of the tableaux obtained from applying \( F^{(k_j-1)}_{r+1,r+2} \) to \( \sum_\alpha T_\alpha \) will be row increasing, but since \( T_j \sim_R T_\alpha \), any \( S \) in the sum \( F^{(k_j-1)}_{r+1,r+2} T_\alpha \) will be row equivalent to some \( T_\beta \) in the sum \( F^{(k_j-1)}_{r+1,r+2} T_j \). Thus, all \( T_{j-1} \triangleright_D W \sim_R T_\beta \sim_R S \) for some \( \lambda \)-tableau \( W \). Write

\[
F^{(k_0)}_{\ell-1,\ell} F^{(k_1)}_{\ell-2,\ell-1} \cdots F^{(k_j)}_{r,r+1} T(\lambda) = F^{(k_0)}_{\ell-1,\ell} T(0) + \sum_\alpha a_\alpha T_\alpha,
\]

where each weakly row increasing \( T_\kappa \) in the sum \( \sum_\kappa a_\kappa T_\kappa \) has \( T_0 \triangleright_D T_\kappa \) and all other tableaux in the sum are row equivalent to a tableau of that sort. Now, \( F^{(k_0)}_{\ell-1,\ell} T_0 = T + \sum_\alpha c_\alpha T_\alpha \) where every weakly row increasing \( T_\alpha \) has \( T \triangleright_D T_\alpha \) and the other tableaux are row equivalent to one of these, using the same argument as above.

For the remaining tableaux \( T_\kappa \) in the sum (9), consider first those \( T_\kappa \) that are weakly row increasing. Since every entry in \( T_0 \) that sits above row \( \ell - 1 \) is less than or equal to \( \ell - 1 \) and \( T_0 \triangleright_D T_\kappa \), every entry in row \( \ell - 1 \) of \( T_\kappa \) is equal to \( \ell - 1 \). Thus, there are exactly \( k_0 \) entries equal to \( \ell - 1 \) above row \( \ell - 1 \). One tableau that arises from applying \( F^{(k_0)}_{\ell-1,\ell} \) to \( T_\kappa \) is the tableau \( S \) where \( S \) comes from changing the \( k_0 \) entries equal to \( \ell - 1 \) above row \( \ell - 1 \) in \( T_\kappa \) to \( \ell \)'s; certainly \( S \) is weakly row increasing.

Then, if \( T_0 \triangleright_D V_1 \triangleright_D \cdots \triangleright_D V_r \triangleright_D T_\kappa \), we have \( T \triangleright_D V_1' \triangleright_D \cdots \triangleright_D V_r' \triangleright_D S \) where \( V_i' \) comes from \( T \) by swapping the entries in the same boxes that were swapped to get from \( T_1 \) to \( V_i \) (or not at all if this would mean swapping two entries equal to \( \ell \)), etc. One may also obtain weakly row increasing tableaux from \( T_\kappa \) by using \( F^{(k_0)}_{\ell-1,\ell} \) to change some \( \ell - 1 \)'s in row \( \ell - 1 \) to \( \ell \)'s and some above row \( \ell - 1 \) to \( \ell \)'s. Suppose that \( V \) is such a tableau. We have \( S \triangleright_D V \), by exchanging a series of \( \ell - 1 \)'s with \( \ell \)'s, so \( T \triangleright_D S \triangleright_D V \). If \( F^{(k_0)}_{\ell-1,\ell} \) is applied to a non-weakly row increasing \( T_{\kappa'} \) from the \( \sum_\kappa a_\kappa T_\kappa \) portion of the sum (9), then, since there is a weakly row increasing \( T_\kappa \) in the sum with \( T_\kappa \sim_R T_{\kappa'} \), any tableau \( V \) in the sum \( F^{(k_0)}_{\ell-1,\ell} T_{\kappa'} \) will also satisfy \( T \triangleright_D W \sim_R V \) for some tableau \( W \). Thus, all non-weakly row increasing tableaux in the sum \( F^{(k_0)}_{\ell-1,\ell} (T_0 + \sum_\alpha a_\alpha T_\alpha) \) are either row equivalent to \( T \) or row equivalent to a tableau \( W \) with \( T \triangleright_D W \). This shows that, for \( T \) with weight \( \chi \) with \( \chi_\ell > \lambda_\ell \) and \( \chi_\ell = \lambda_i \) for \( i \neq \ell \) we have \( a(T)T(\lambda) = T + \sum_\alpha c_\alpha T_\alpha \) where for each \( c_\alpha \neq 0 \) we have \( T \triangleright_D W \sim_R T_\alpha \) for some tableau \( W \).
For an arbitrary \(T\), suppose that \(\ell\) is the smallest entry in \(T\) that is in a row \(x\) with \(x < \ell\) and suppose that \(r\) is the smallest row that contains an \(\ell\) with \(r < \ell\). Let \(T_0\) be the tableau that comes from \(T\) by replacing all \(\ell\)'s above row \(\ell\) with the row number to which they belong. Then \(a(T)T(\lambda) = F_{\ell-1,\ell}^{(k_0)} \cdots F_{r,r+1}^{(k_i)} a(T_0)T(\lambda)\) so by induction we have \(a(T)T(\lambda) = F_{\ell-1,\ell}^{(k_0)} \cdots F_{r,r+1}^{(k_i)} (T_0 + \sum_{\beta} a_\beta T_\beta)\), where each \(T_\beta\) has \(T_0 \supseteq_D W \sim_R T_\beta\).

There are no entries equal to \(x\) above row \(x\) for any \(x\) with \(r \leq x \leq \ell\) in \(T\), so the same is true of \(T_0\). Thus, an argument similar to that above shows that \(F_{\ell-1,\ell}^{(k_0)} \cdots F_{r,r+1}^{(k_i)} T_1 = T_J + \sum_{\alpha} c_\alpha T_\alpha\) where \(T \supseteq_D W \sim_R T_\alpha\) for some tableau \(W\), for each \(c_\alpha \neq 0\) in the sum.

We can also use a similar argument to that above to prove that, for each weakly row increasing \(T_\beta\) in the sum above, \(F_{\ell-1,\ell}^{(k_0)} \cdots F_{r,r+1}^{(k_i)} T_\beta = \sum_i a_i T_i\), where each \(T_i\) has \(T \supseteq_D W \sim_R T_i\). Since the other tableaux in the sum are row equivalent to weakly row increasing tableaux, we obtain the result for all \(T_\beta\) in the sum. □

From the above two lemmas and (7), we obtain the following corollary.

**Corollary 7.4.** Suppose that \(T\) is a semistandard \(\lambda\)-tableau. Then

\[
\theta(a(T))\xi_{I(\lambda),I(\lambda)} = \sum_{Q \in R(\lambda,n)} a_Q \xi_{Q,I(\lambda)},
\]

where \(a_Q = q^{r(T) - s(T)} \langle v_{I(\lambda)}, b(T) v_Q \rangle\) and, if \(a_Q \neq 0\), then \(T \supseteq_D T_{Q'}\).

In light of the above, we can obtain a linear combination

\[
a(T)w_\lambda = \theta(a(T))\xi_{I(\lambda),I(\lambda)} w_\lambda = \sum_{Q \in R(\lambda,n)} a_Q \xi_{Q,I(\lambda)} w_\lambda
\]

by considering all \(T_{Q'} \in RT(\lambda,n)\) with \(T \supseteq_D T_{Q'}\). If for any \(Q\) in this sum \(T_{Q'}\) is not semistandard, rewrite \(\xi_{Q,I(\lambda)} w_\lambda\) as a linear combination of basis elements \(\xi_{J,I(\lambda)} w_\lambda\), where each \(J\) from the sum gives a semistandard \(\lambda\)-tableau \(T_J\). We can then use the algorithm from [13] to write each global crystal basis vector as a linear combination of vectors from the Leclerc-Toffin basis, which in turn gives a linear combination of elements from the \(q\)-Schur algebra version of the Carter-Lusztig basis.

**Example 7.5.**

1. Let \(\lambda = (2,1)\). The weakly row increasing \(\lambda\)-tableaux with weight \(\chi = (1,1,1)\) are \(T_1 = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\), \(T_2 = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\), \(T_3 = \begin{array}{c} 2 \\ 3 \\ 1 \end{array}\).

In this case, \(a(T_1)w_\lambda \equiv \omega(T_1) \mod qL_W(\lambda)\) and \(a(T_2)w_\lambda \equiv \omega(T_2) \mod qL_W(\lambda)\) so these are the global basis vectors for the weight space \(V(\lambda)^{\chi}\), where \(\chi = (1,1,1)\).
We have $a(T_2)w_\lambda = \xi_{(1,2,3),(1,1,2)}w_\lambda$ and $a(T_1)w_\lambda = (\xi_{(1,3,2),(1,1,2)} + a_1\xi_{(1,2,3),(1,1,2)})w_\lambda$. Computing $B(T_1)v_1 \otimes v_1 \otimes v_2$ and employing Corollary 7.4 yields $a_1 = q^{-1}$. Compare with Example 5.11.

2. Let $\lambda = (3,1)$ and consider the weakly row increasing $\lambda$-tableaux of weight $\chi = (1,2,1,0)$: $T_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, $T_2 = \begin{bmatrix} 1 & 2 & 2 \\ 3 \\ 1 \end{bmatrix}$, $T_3 = \begin{bmatrix} 2 & 2 & 3 \\ 1 \end{bmatrix}$. Let $G(T_1)$ and $G(T_2)$ denote the global crystal basis vectors for the weight space $V(\lambda)^\chi$.

We have $a(T_2)w_\lambda = \xi_{(1,2,3),(1,1,2)}w_\lambda = G(T_2)$ and, since $T_1 \triangleright D T_2$, $a(T_1)w_\lambda = q^{-1}\xi_{(1,2,3,2),(1,1,1,2)} + a_1\xi_{(1,2,3),(1,1,1,2)}$, $a \in \mathbb{Q}[q, q^{-1}]$. But

$$q^{r(T_2)-s(T_2)}\langle v_{T_1(\lambda)}, B(T_1)v_1 \otimes v_2 \otimes v_2 \otimes v_3 \rangle = q^2(q^{-2} + q^{-4}),$$

so

$$a(T_1)w_\lambda = q^{-1}\xi_{(1,2,3,2),(1,1,1,2)}w_\lambda \equiv (1 + q^{-2})\xi_{(1,2,3,2),(1,1,1,2)}w_\lambda \mod qL_W(\lambda),$$

we have $a(T_1)w_\lambda \equiv \omega(T_1) \mod qL_W(\lambda)$, $a(T_1)w_\lambda$ is not a global crystal basis vector. However, since $a(T_1)w_\lambda - a(T_2)w_\lambda \equiv \omega(T_1) \mod qL_W(\lambda),$ we have

$$G(T_1) = a(T_1)w_\lambda - a(T_2)w_\lambda = q^{-1}\xi_{(1,2,3,2),(1,1,1,2)}w_\lambda + q^{-1}\xi_{(1,2,3,2),(1,1,1,2)}w_\lambda.$$ 

It follows that

$$\{\xi_{(1,2,3,2),(1,1,1,2)}w_\lambda, q^{-2}\xi_{(1,2,3,2),(1,1,1,2)}w_\lambda + q^{-1}\xi_{(1,2,3,2),(1,1,1,2)}w_\lambda\}$$

is the portion of the global crystal basis for the weight space $V(\lambda)^\chi$.

**References**


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