Abstract. We give lattice path proofs of determinantal formulas for orthosymplectic characters. We use the \(spo(2m, n)\)-tableaux introduced by Benkart, Shader and Ram, which have both a semistandard symplectic part and a row-strict part. We obtain orthosymplectic Jacob-Trudi identities and an orthosymplectic Giambelli identity by associating \(spo(2m, n)\)-tableaux to certain families of nonintersecting lattice paths and using an adaptation of the Gessel-Viennot method.


e 1. Introduction

Given a partition \(\lambda\), the Jacobi-Trudi identity gives a determinantal expression for the Schur function \(s_\lambda(x_1, \ldots, x_n)\) in terms of homogenous symmetric polynomials and the dual Jacobi-Trudi identity gives an expression for \(s_\lambda(x_1, \ldots, x_n)\) in terms of elementary symmetric functions (see [9]). The Giambelli identity gives a determinantal expression for \(s_\lambda(x_1, \ldots, x_n)\) in terms of Schur functions associated to the principal hooks of the appropriate Young diagram of shape \(\lambda\). Gessel and Viennot gave lattice path proofs of the Jacobi-Trudi identities by interpreting semistandard \(\lambda\)-tableaux as certain families of nonintersecting lattice paths [7] and Stembridge gave a lattice path proof of the Giambelli identity [11].

Semistandard symplectic tableaux index irreducible representations of symplectic groups and symplectic Schur functions can be described in terms of the semistandard symplectic tableaux of King [8]. Fulmek and Krattenthaler [5] proved symplectic and orthogonal Jacobi-Trudi and Giambelli identities for symplectic Schur functions and orthogonal Schur functions in [5] by interpreting semistandard symplectic tableaux as certain families of nonintersecting lattice paths.

Moving to the superalgebra setting, we have the supersymmetric Schur functions (or hook Schur functions), which are characters of irreducible representations of the Lie superalgebras \(gl(m|n)\) [3]. Berele and Regev gave combinatorial descriptions of supersymmetric Schur functions in terms of hybrid Schur functions involving tableaux with both a column-strict part and a row-strict part. Goulden and Greene gave lattice path proofs of determinantal formulas for supersymmetric Schur functions in [6].

Balantekin and Bars gave Jacobi-Trudi formulas for characters of irreducible representations of orthosymplectic Lie algebras \(spo(2m, n)\) in [1], which were

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proved using a different approach in [2]. Combinatorial descriptions of $spo(2m, n)$-characters were first given in [2] using $spo(2m, n)$-tableaux, which have both a symplectic part (in the sense of King [8]) and a row-strict part.

The aim of this paper is to give lattice path proofs of Jacobi-Trudi formulas and Giambelli formulas for characters of representations of orthosymplectic Lie superalgebras $spo(2m, n)$ using $spo(2m, n)$-tableaux.

In Section 3, we give lattice path proofs of the symplectic Jacobi-Trudi formulas and Giambelli formulas, using different families of paths than those used in [5] and some algebraic identities. Since orthosymplectic Schur functions can be described using tableaux that are hybrid symplectic Schur functions and classical Schur functions, our approach in Section 3 leads into that used in Section 4, where we give lattice path proofs of orthosymplectic determinantal identities.

2. Preliminaries

A partition is a $k$-tuple of integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. The length of $\lambda$ is $\ell(\lambda) = k$. The Young diagram of shape $\lambda$ consists of $N = \sum_{i=1}^{k} \lambda_i$ boxes in $k$ left-justified rows with $\lambda_i$ boxes in the $i$th row. The conjugate of $\lambda$ is the partition $\lambda^t = (\lambda_1^t, \lambda_2^t, \ldots, \lambda_k^t)$ where $\lambda_i^t$ denotes the number of boxes in the $i$th column of the Young diagram of shape $\lambda$. A $\lambda$-tableau is a filling the Young diagram of shape $\lambda$ with entries from a set $\{1, 2, \ldots, n\}$ of positive integers. A $\lambda$-tableau is semistandard if the entries in the rows are weakly increasing from left to right and the entries in the columns are strictly increasing from top to bottom.

For a $\lambda$-tableau $T$, let $a_i(T)$ denote the number of entries equal to $i$ in $T$. The weight of $T$ is the monomial in the variables $X = \{x_1, x_2, \ldots, x_n\}$ defined by $\text{wt}(T) = \prod_{i=1}^{n} x_i^{a_i(T)}$. The Schur function corresponding to $\lambda$ is $s_\lambda(X) = \sum_T \text{wt}(T)$, where the sum runs over all semistandard $\lambda$-tableaux $T$ with entries in $\{1, 2, \ldots, n\}$.

Define the $r$th homogeneous symmetric function by $h_r(X) = \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq n} x_{i_1} \cdots x_{i_r}$, and the $r$th elementary symmetric function by $e_r(X) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} x_{i_1} \cdots x_{i_r}$, where both $h_r(X) = 0$ and $e_r(X) = 0$ if $r < 0$ and $h_0(X) = e_0(X) = 1$.

For a partition $\lambda$ with $\ell(\lambda) = k \leq n$, we have the Jacobi-Trudi identity:

$$s_\lambda(X) = |h_{\lambda_i-i+j}(X)|_{k \times k},$$

and the dual Jacobi-Trudi identity

$$s_\lambda(X) = |e_{\lambda_i-i+j}(X)|_{\lambda_1 \times \lambda_1}.$$

The $i$th principal hook of a $\lambda$-tableau $T$ is the Young diagram consisting of the $(i, i)$-box together with all boxes in the $i$th row that are right of the $(i, i)$-box and all boxes in the $i$th column that are below the $(i, i)$-box. If $T$ has $r$ nonempty
principal hooks, then \( T \) is said to have rank \( r \). For \( 1 \leq i \leq r \), let \( \alpha_i = \lambda_i - i \) and \( \beta_i = \lambda_i^t - i \) and let \( \alpha = (\alpha_1, \ldots, \alpha_r) \) and \( \beta = (\beta_1, \ldots, \beta_r) \). Then \( \lambda = (\alpha|\beta) \) is the Frobenius notation for the partition \( \lambda \). Using the Frobenius notation, we have the Giambelli identity:

\[
s_{(\alpha|\beta)}(X) = |s_{(\alpha|\beta)}(X)|_{r \times r}.
\]

Proofs of the above three identities can be found in [9].

Let \( \lambda \) be a partition with \( \ell(\lambda) \leq m \) and consider the set \( \{1, \bar{1}, \ldots, m, \bar{m}\} \) with the ordering \( 1 < \bar{1} < \cdots < m < \bar{m} \). A semistandard symplectic \( \lambda \)-tableau (see [8]) is a \( \lambda \)-tableau \( T \) with entries from \( \{1, \bar{1}, 2, \bar{2}, \ldots, m, \bar{m}\} \) that is semistandard with respect to the ordering and satisfies the additional property (symplectic condition) that the entries in the \( i \)th row of \( T \) are greater than or equal to \( i \) for each \( 1 \leq i \leq m \).

Define the weight of a symplectic \( \lambda \)-tableau \( T \) by \( \text{wt}(T) = \prod_{i=1}^{m} x_{i, T}^{a_i(T) - s_i(T)} \) where \( a_i(T) \) is equal to the number of entries equal to \( i \) that appear in \( T \) and let \( X = \{x_1, x_1^{-1}, \ldots, x_m, x_m^{-1}\} \). The symplectic Schur function is defined by

\[
s_{p_{\lambda,2m}}(X) = \sum_T \text{wt}(T),
\]

where the sum runs over the semistandard symplectic \( \lambda \)-tableaux with entries in \( X \).

**Example 2.1.** Let \( \lambda = (2, 2, 1) \). The \( \lambda \)-tableau \( T \) below is not semistandard symplectic since it contains a violation of the symplectic condition while the \( \lambda \)-tableau \( S \) is semistandard symplectic:

\[
T = \begin{array}{ccc}
1 & 2 & 2 \\
\bar{1} & \bar{2} & 1 \\
3 & & \\
\end{array}, \quad
S = \begin{array}{ccc}
1 & 2 & 2 \\
2 & \bar{1} & 1 \\
3 & & \\
\end{array}
\]

By listing the five semistandard symplectic tableaux of shape \( \lambda = (1, 1) \) with entries in the set \( \{1, \bar{1}, 2, \bar{2}\} \), it can be seen that the symplectic Schur function corresponding to \( \lambda = (1, 1) \) and \( m = 2 \) is

\[
s_{p_{\lambda,2m}}(x_1, x_1^{-1}, x_2, x_2^{-1}) = x_1x_2 + x_1x_2^{-1} + x_1^{-1}x_2 + x_1^{-1}x_2^{-1} + 1.
\]

We will give proofs of symplectic and orthosymplectic determinantal formulas using lattice paths. Let \( \pi = p_0 \cdots p_n \) be a sequence of lattice points in \( \mathbb{Z} \times \mathbb{Z} \) such that \( p_{k+1} = p_k + s_k \) for \( 0 \leq k < n \) where \( s_k \in S \) and \( S \) is a fixed subset of \( \mathbb{Z} \times \mathbb{Z} \) that does not contain \((0,0)\). We say that \( \pi \) is a (lattice) path with steps restricted to \( S \) that has initial point \( p_0 \), terminal point \( p_n \) and length \( n \). Two paths will be said to intersect if they have at least one lattice point in common.

A family of paths with initial points \( P_1, \ldots, P_k \) and terminal points \( Q_1, \ldots, Q_k \) is a \( k \)-tuple \( \epsilon = (\pi_1, \ldots, \pi_k) \) where, for each \( i \), \( \pi_i \) is a path with initial point \( P_i \) and endpoint \( Q_{\sigma(i)} \) for some \( \sigma \in S_k \), the symmetric group on \( k \) letters. A family of paths is said to be intersecting if at least two paths in the family intersect and nonintersecting otherwise.
3. Symplectic Determinantal Formulas

In this section, we develop symplectic determinantal formulas by adapting the Gessel-Viennot method. These identities were also proved in [5] using different families of lattice paths. We build on the approach we use here to prove orthosymplectic determinantal formulas in the next section.

Along the \( x \)-axis, label the vertical lines \( x = 1, x = 2, \) et cetera, up to \( x = 2m \) consecutively with the symbols \( 1, \ mat 1, 2, \ mat 2, \ldots, m, \ mat m. \) Let \( l(\lambda) = k \) and let \( P_i = (2i - 1, k - i) \) and \( Q_i = (2m, \lambda_i + k - i) \), where \( 1 \leq i \leq k \). There is a bijection between the set of semistandard symplectic \( \lambda \)-tableaux and families of nonintersecting lattice paths with steps restricted to \((1, 0)\) (horizontal) and \((0, 1)\) (vertical) with initial and terminal points given by the \( P_i \) and \( Q_i \). We associate a tableau to such a family by interpreting the \( i \)th row of \( T \) as a path from \( P_i \) to \( Q_i \): for each \( a \in \{1, \ mat 1, \ldots, m, \ mat m\} \), place a vertical step at the \( x \)-coordinate labelled \( a \) for each occurrence of \( a \) in the \( i \)th row of \( T \) and fill in the gaps with horizontal steps to complete the path. The inverse map is given by reversing the procedure.

Given a path \( \pi \) from \( P_i \) to \( Q_j \), with steps restricted to \((1, 0)\) and \((0, 1)\), assign the weight \( x_a \) to each vertical step with \( x \)-coordinate labelled \( a \) and the weight \( x_a^{-1} \) to each vertical step with \( x \)-coordinate labelled \( \overline{a} \). For each horizontal step assign a weight of one. The weight \( wt(\pi) \) of the path \( \pi \) is the product of the weights of the individual steps. Given a family of paths \( \epsilon = (\pi_1, \pi_2, \ldots, \pi_k) \), where \( \pi_i \) has initial point \( P_i \) and terminal point \( Q_{\sigma(i)} \) for each \( i, \sigma \in S_k \), let

\[
(1) \quad wt(\epsilon) = \prod_i wt(\pi_i) \quad \text{and} \quad (-1)^\epsilon = sgn(\sigma).
\]

**Example 3.1.** Let \( T = \begin{array}{cccc}
1 & 1 & 2 & 3 \\
2 & 3 & 3 & 3 \\
4 & 4 & 4 & 3 \\
\end{array} \). The family of nonintersecting lattice paths corresponding to the rows of \( T \) is given below.

![Diagram of nonintersecting lattice paths](image)

Let \( \pi_i \) denote the path from \( P_i \) to \( Q_i \), for \( 1 \leq i \leq 4 \) and let \( \epsilon = (\pi_1, \pi_2, \pi_3, \pi_4) \). Then \( wt(\pi_1) = x_1 x_4^{-1} x_2 x_3 = x_2 x_3 \) and \( wt(\epsilon) = x_2^2 \).
Theorem 3.2. Let $\lambda$ be a partition of $n$. Then
\[ sp_{\lambda,2m}(X) = |h_{\lambda_j}^{-j+i}(x_i, x_i^{-1}, \ldots, x_m, x_m^{-1})|. \]

Proof. We briefly outline the proof, which is a straightforward adaptation of the Gessel-Viennot method [7] that was used to prove the classical Jacobi-Trudi formula (see [10, Theorem 4.5.1]). Let $\epsilon = (\pi_1, \pi_2, \ldots, \pi_k)$ be a family of paths with initial points given by the points $P_i = (2r - 1, k - i)$, $1 \leq i \leq k$, and terminal points given by the points $Q_i = (2m, \lambda_i + k - i)$, $1 \leq i \leq k$, and steps restricted to $(1, 0)$ and $(0, 1)$. Then $h_{\lambda_j-j+i}(x_i, x_i^{-1}, \ldots, x_m, x_m^{-1}) = \sum_{P_i \rightarrow Q_j} \text{wt}(\pi)$, where the sum runs over all paths from $P_i$ to $Q_j$ and
\[ |h_{\lambda_j-j+i}(x_i, x_i^{-1}, \ldots, x_m, x_m^{-1})| = \sum_{\epsilon} (-1)^{\text{wt}(\epsilon)}, \]
where the sum runs over all families of paths with initial and terminal points given by the $P_i$ and $Q_j$, $1 \leq i, j \leq k$. The terms in the righthand side of the above equation that correspond to intersecting families cancel in pairs: given an intersecting family $\epsilon = (\pi_1, \pi_2, \ldots, \pi_k)$, let $\pi_i$ be the least index $i$ for which $\pi_i$ intersects another path and let $A$ be the first intersection point on the path and $\pi_j$ the path of next lowest index that passes through $A$. Interchange the portions of $\pi_i$ and $\pi_j$ that occur after $A$ to get two new paths $\pi'_i$ and $\pi'_j$ that also intersect at $A$. The new family $\epsilon'$, given by replacing $\pi_i$ with $\pi'_i$ and $\pi_j$ with $\pi'_j$, has the opposite sign as $\epsilon$ and this procedure defines a weight-preserving involution on the set of intersecting lattice paths so the righthand sum runs over all nonintersecting families $\epsilon = (\pi_1, \pi_2, \ldots, \pi_k)$ where the initial point of $\pi_i$ is $P_i$ and the terminal point is $Q_i$. Since there is a bijection between the set of semistandard symplectic $\lambda$-tableaux and such families, the result follows. \hfill \Box

To relate our version of the Jacobi-Trudi formula above to that proved in [5], we need the following two Lemmas. Define $h^{(p)}_r = h_r(x_p, x_p^{-1}, x_{p+1}, x_{p+1}^{-1}, \ldots, x_m, x_m^{-1})$ for $p \geq 1$ and note that $h^{(1)}_r = h_r(X)$. Our first Lemma follows easily from the definition of homogeneous polynomials.

Lemma 3.3. Let $p \geq 1$ and $r \geq 2$. Then $h^{(p+1)}_r = h^{(p)}_r - (x_p + x_p^{-1}) h^{(p)}_{r-1} + h^{(p)}_{r-2}.$

Lemma 3.4. Let $p \geq 1$ and let $A = (h^{(p+i-1)}_{r_j})_{1 \leq i, j \leq n}$. Then
\[ |A| = |h^{(p)}_{r_j} : h^{(p)}_{r_j+i-1} + h^{(p)}_{r_j-(i-1)}|, \]
where $h^{(p)}_{r_j}$ gives the first row of the matrix for $1 \leq j \leq n$, and subsequent rows are given by $h^{(p)}_{r_j+i-1} + h^{(p)}_{r_j-(i-1)}$ for $2 \leq i \leq n$ and $1 \leq j \leq n$.

Proof. Expand the determinant of $A$ across the first row to get
\[ |A| = \sum_{k=1}^n (-1)^{k+1} h^{(p)}_{r_k} M_{1k}, \]
Corollary 3.5. \( |A| = |h_{r_j}^{(p)} : h_{r_j+1}^{(p+1)} : h_{r_j+i-1}^{(p+1)} + h_{r_j+i+3}^{(p+1)}| \) 

where 3 \( \leq i \leq n \), 1 \( \leq j \leq n \). (Here the first and second rows of the matrix are given by \( h_{r_j}^{(p)} \) and \( h_{r_j+1}^{(p+1)} \), respectively, for 1 \( \leq j \leq n \), and subsequent rows are given by \( h_{r_j+i-1}^{(p+1)} + h_{r_j+i+3}^{(p+1)} \), for 1 \( \leq j \leq n \).) 

By Lemma 3.3, \( h_{r_j+1}^{(p+1)} = h_{r_j+1}^{(p)} - (x_p + x_p^{-1}) h_{r_j}^{(p)} + h_{r_j+1}^{(p)} \), so 

\[ |A| = |h_{r_j}^{(p)} : h_{r_j+1}^{(p)} : h_{r_j+i-1}^{(p+1)} + h_{r_j+i+3}^{(p+1)}|. \]

For \( i \geq 3 \), \( h_{r_j+i-1}^{(p+1)} + h_{r_j+i+3}^{(p+1)} \) is equal to 

\[ h_{r_j+i-1}^{(p)} - (x_p + x_p^{-1}) (h_{r_j+i-2}^{(p)} + h_{r_j+i+2}) + h_{r_j+i-3}^{(p)} + h_{r_j+i+3}^{(p)} + h_{r_j-(i-1)}^{(p)}, \]

and since the middle terms give linear combinations of previous rows, by induction we have \( |A| = |h_{r_j}^{(p)} : h_{r_j+i-1}^{(p+1)} + h_{r_j-(i-1)}^{(p)}| \). \( \Box \)

We obtain the Jacobi-Trudi identity proved in [5, (3.9)] as a consequence of Lemma 3.4 with \( p = 1 \) and \( r_j = \lambda_j - j + 1 \) for 1 \( \leq j \leq n \).

Corollary 3.5. Let \( \lambda \) be a partition. Then 

\[ sp_{\lambda,2m}(X) = |h_{\lambda_j-j+1}(X) : h_{\lambda_j-j+i}(X) + h_{\lambda_j-j-i+2}(X)|. \]

We can also obtain a determinantal expression for \( sp_{\lambda,2m} \) involving elementary symmetric functions by considering the conjugate partition. There is a bijection between the set of semistandard symplectic \( \lambda \)-tableaux and nonintersecting families of lattice paths \( \epsilon = (\pi_1, \ldots, \pi_k) \) with initial and terminal points given by \( P_i = (0, -i) \) and \( Q_i = (2m, \lambda'_i - i) \), 1 \( \leq i \leq \lambda_1 \), and steps restricted to (1,0) (horizontal) and (1,1) (diagonal) for which the first path \( \pi_1 \) does not intersect the line \( y = \frac{1}{2}x \). Label the vertical lines \( x = 0, x = 1, \ldots, x = 2m - 1 \) consecutively by the symbols 1, \( \overline{1} \), 2, \( \overline{2} \), \ldots, \( m \), \( \overline{m} \). A family of paths is obtained by associating the \( i \)-th column of a semistandard symplectic \( \lambda \)-tableau \( T \) to a path from \( P_i = (0, -i) \) to \( Q_i = (2m, \lambda'_i - i) \), where a diagonal step starting at \( a \) (respectively \( \overline{a} \)) indicates that \( a \) belongs to the \( i \)-th column and a horizontal step starting at \( a \) indicates that \( a \) does not appear in the \( i \)-th column. A family of nonintersecting paths \( \epsilon = (\pi_1, \pi_2, \ldots, \pi_{\lambda_1}) \) with initial and terminal points \( P_i \) and \( Q_i \) intersects \( y = \frac{1}{2}x \) if and only if \( \pi_1 \) intersects this line at the point \((2a, a)\) in the \( xy \)-plane for some \( a \) with 1 \( \leq a \leq m \). Then there is a diagonal step starting at \( \overline{a} \) and since a diagonal steps occur in the path prior to this step, there are \( a \) entries above the \( \overline{a} \) in the \( i \)-th column of \( T \) which forces an \( \overline{a} \) in row \( a + 1 \) of the first column of the associated tableau—a violation of the symplectic condition. It follows that \( \pi_1 \) intersects \( y = \frac{1}{2}x \) if and only if the tableau given by \( \epsilon = (\pi_1, \pi_2, \ldots, \pi_k) \) violates the symplectic condition.
Given a path $\pi$ from $P_i = (0, -i)$ to $Q_j = (2m, \lambda_j^t - j)$, assign a weight $x_a$ to each diagonal step starting at $a$ and $x_a^{-1}$ to each diagonal step starting at $\overline{a}$ and define $\text{wt}(\pi)$ and $\text{wt}(\epsilon)$, for a family of paths, as in (1).

Example 3.6. The family of paths associated to the columns of the tableau $T$ from Example 3.1 is as follows.

Our proof of the symplectic dual Jacobi-Trudi identity is less straightforward and relies on constructing a certain reflected path when a path intersects $y = \frac{1}{2}x$. Given a path $\pi$ with initial point $P_i = (0, -i)$ and terminal point $Q_j = (2m, \lambda_j^t - j)$ that intersects $y = \frac{1}{2}x$, form a new path $\pi'$ beginning at $P'_i = (0, i)$ in the following way. Let $A$ be the western-most point at which $\pi$ intersects $y = \frac{1}{2}x$. For all steps west of $A$, replace a diagonal step in $\pi$ with a horizontal step at $a$ in $\pi'$ (respectively $a$) if and only if there is also a diagonal step at $a$ in $\pi$ (respectively $a$). Similarly, replace a horizontal step at $a$ in $\pi$ with a diagonal step at $a$ in $\pi'$ (respectively $\overline{a}$) if and only if there is also a horizontal step at $a$ in $\pi$ (respectively $a$). Otherwise, place the same type of step at $a$ (respectively $\overline{a}$) in $\pi'$. The portion of the path that lies east of $A$ is the same in $\pi'$ as in $\pi$. (See Example 3.7.)

If a path $\pi$ with initial point $P_i = (0, -i)$ intersects $y = \frac{1}{2}x$ at $A = (2a, a)$, there are $a + i$ diagonal steps prior to $A$ and these occur at values on the $x$-axis labelled by $1, 1, 2, \ldots, a, \overline{a}$. Thus, there are at least $i$ elements $b, 1 \leq b \leq a$, with diagonal steps at both $b$ and $\overline{b}$; suppose there are $N + i$ such elements, which contribute $2N + 2i$ diagonal steps in $\pi$ and $2N + 2i$ horizontal steps in $\pi'$. There must be $N$ elements $b$ with $1 \leq b \leq a$ such that there are horizontal steps at both $b$ and $\overline{b}$ in $\pi$, which contribute $2N$ diagonal steps to $\pi'$. It follows that $\pi$ has exactly $2i$ more diagonal steps than $\pi'$ and since there are $\lambda_j^t - j + i$ diagonal steps in $\pi$, there are $\lambda_j^t - j - i$ diagonal steps in $\pi'$, which shows that $Q_j = (2m, \lambda_j^t - j)$ is the terminal point of $\pi'$. It is clear that $\text{wt}(\pi) = \text{wt}(\pi')$.

This is an involution so it gives a weight-preserving bijection from the family $Q$ of paths with initial and terminal points $P_i$ and $Q_j$ that intersect $y = \frac{1}{2}x$ and the family $Q'$ of paths from $P'_i$ to $Q_j$. 
**Example 3.7.** Consider the path $\pi$ from $P_1 = (0, -1)$ to $Q_1 = (10, 4)$ in the first diagram below. It first intersects $y = \frac{1}{2}x$ (labelled $\ell$ in the diagrams) at the point $A = (6, 3)$. Our construction gives the path $\pi'$ from $P'_1 = (0, 1)$ to $Q_1$.

The following theorem is also proved in [5, (3.10)].

**Theorem 3.8.** Let $\lambda$ be a partition of $n$. Then

$$sp_{\lambda, 2m}(X) = |e_{\lambda'_{j+i}}(X) - e_{\lambda'_{j-i}}(X)|.$$  

**Proof.** For $1 \leq i \leq \lambda'_1$, let $P_i = (0, -i)$ and $Q_i = (2m, \lambda'_i - i)$. Using the weight-preserving involution described above, we have

$$\sum_{P_i \rightarrow^{\alpha} Q_j} \text{wt}(\alpha) = \sum_{P'_i \rightarrow^{\alpha'} Q_j} \text{wt}(\alpha') = e_{\lambda'_{j-i}}(X),$$

where the first sum runs over the paths $\alpha$ from $P_i$ to $Q_j$ that intersect $y = \frac{1}{2}x$. It follows that

$$\sum_{P_i \rightarrow Q_j} \text{wt}(\pi) = e_{\lambda'_{j+i}}(X) - e_{\lambda'_{j-i}}(X),$$

where the sum runs over all paths $\pi$ from $P_i$ to $Q_j$ that do not intersect $y = \frac{1}{2}x$. Then

$$|e_{\lambda'_{j+i}}(X) - e_{\lambda'_{j-i}}(X)| = \sum_{\epsilon} (-1)^{\epsilon} \text{wt}(\epsilon),$$

where the latter sum runs over the families of paths $\epsilon$, with initial and terminal points given by the $P_i$ and $Q_j$, that do not contain paths that intersect $y = \frac{1}{2}x$.

Using an argument similar to that used in the proof of Theorem 3.2, it can be shown that the intersecting families $\epsilon$ in the right-hand side of the above equation cancel in pairs. Since there is a bijection between semistandard symplectic $\lambda$-tableaux and families of nonintersecting paths $\epsilon$ that do not contain paths that intersect $y = \frac{1}{2}x$, $sp_{\lambda, 2m}(X) = |e_{\lambda'_{j+i}}(X) - e_{\lambda'_{j-i}}(X)|$. \[\square\]

There is also a Giambelli determinant associated with symplectic Schur functions that can be developed using the Gessel-Viennot method. To prove such a formula, we will work with paths that first go in a northwesterly direction, with steps terminating in the region $\{(x, y) \mid y \leq -1\}$ restricted to $(0, 1)$ and $(-1, 0)$, and then in a northeasterly direction, where steps terminating in the region $\{(x, y) \mid y > -1\}$ are restricted to $(1, 0)$ and $(1, 1)$. 
Let \( \lambda = (\alpha_1, \ldots, \alpha_r \mid \beta_1, \ldots, \beta_r) \) be the Frobenius notation for \( \lambda \). Then there is a bijection between semistandard symplectic tableaux and families of paths \( \epsilon = (\pi_1, \pi_2, \ldots, \pi_r) \) (with step restrictions as above), where \( \pi_i \) has initial and terminal points \( P_i = (2m - 1, -1 - \alpha_i) \) and \( Q_i = (2m, \beta_i) \) respectively, and \( \pi_1 \) does not intersect the line \( y = \frac{1}{2}x \). The bijection is given by associating the \( i \)th principal hook of a tableau \( T \) with a path from \( P_i \) to \( Q_i \).

Label the lines \( x = 0, x = 1, \ldots, x = 2m - 1 \) consecutively with the symbols 1, 1, 2, 2, \ldots, \( m \), \( m \) and suppose that the entry in the \( (i, i) \)-box is equal to \( a \) (or \( a \)). Then the northwesterly section of \( \pi_i \) will be the path from \( P_i \) to the point with \( x \)-coordinate label \( a \) (or \( a \)) and \( y \)-coordinate \( -1 \) (call this point \( R_i \)) which has a vertical step on the line with \( x \)-coordinate label \( b \) (or \( b \)) for each occurrence of \( b \) (or \( b \)) that occurs to the right of the \( (i, i) \)-box in the \( i \)th principal hook of \( T \). There is a unique way to place these vertical steps and connect them with horizontal steps, directed west. The northeasterly section of \( \pi_i \) will begin with a diagonal step at \( R_i \), directed east (which is associated with the entry in the \( (i, i) \)-box of \( T \)), and will have a diagonal step starting at \( b \) (or \( b \)) for each occurrence of \( b \) (or \( b \)) that occurs below the \( (i, i) \)-box in the \( i \)th principal hook of \( T \). There is a unique way to place these diagonal steps and connect them with horizontal steps, directed east to complete the path from \( P_i \) to \( Q_i \).

As in the set up for the dual Jacobi-Trudi formula, a family of paths \( \epsilon = (\pi_1, \pi_2, \ldots, \pi_r) \) intersects the line \( y = \frac{1}{2}x \) if and only if the corresponding tableau contains a violation of the symplectic condition. For a path from \( P_i \) to \( Q_j \), and for a family \( \epsilon = (\pi_1, \pi_2, \ldots, \pi_r) \) with initial and terminal points given by the \( P_i \) and \( Q_j \), we define \( \text{wt}(\pi), \text{wt}(\epsilon) \) and \( (-1)^{\epsilon} \) as in (1).

**Example 3.9.** Consider the \( \lambda \)-tableau \( T \) from Example 3.1. The Frobenius notation for \( \lambda \) is \( \lambda = (3, 2, 0 \mid 3, 1, 0) \) and \( T \) has rank three. The family of paths \( \epsilon = (\pi_1, \pi_2, \pi_3) \) associated to the principal hooks of \( T \) is as follows.

![Diagram of lattice paths](image)

**Theorem 3.10.** Let \( \lambda = (\alpha \mid \beta) \) be the Frobenius notation for a partition \( \lambda \) with rank \( r \). Then

\[
sp_{\lambda, 2m}(X) = \left| sp_{(\alpha_i \mid \beta_j), 2m}(X) \right|_{1 \leq i, j \leq r}.
\]
Proof. Let \( \epsilon = (\pi_1, \pi_2, \ldots, \pi_r) \) be a family of paths (with step restrictions as described above) with initial points given by \( P_i = (2m - 1, -1 - \alpha_i) \) and terminal points given by \( Q_i = (2m, \beta_i) \), \( 1 \leq i \leq r \). Then

\[
sp(\alpha_i|\beta_j),2m(X) = \sum_{P_i \rightarrow Q_j} \text{wt}(\pi),
\]

where the sum runs over all paths from \( P_i \) to \( Q_j \) that do not intersect \( y = \frac{1}{2}x \) so

\[
|sp(\alpha_i|\beta_j),2m(X)|_{1 \leq i,j \leq r} = \sum_{\epsilon} (-1)^{\epsilon} \text{wt}(\epsilon),
\]

where the sum is over all families of paths \( \epsilon = (\pi_1, \pi_2, \ldots, \pi_r) \) with initial points and terminal points given by \( P_i \) and \( Q_j \) that do not intersect \( y = \frac{1}{2}x \). As in the proof of Theorem 3.2, intersecting families \( \epsilon \) in the right hand side sum cancel in pairs so

\[
|sp(\alpha_i|\beta_j),2m(X)|_{1 \leq i,j \leq r} = \sum_{\epsilon} \text{wt}(\epsilon),
\]

where the sum is over all nonintersecting families \( \epsilon = (\pi_1, \pi_2, \ldots, \pi_r) \); in other words, each \( \pi_i \) has initial point \( P_i \) and terminal point \( Q_i \) and \( \pi_1 \) does not intersect \( y = \frac{1}{2}x \). Since such paths are in bijection with semistandard symplectic tableaux, the result follows. \( \square \)

4. Orthosymplectic Determinantal Formulas

In this section we derive determinantal formulas for orthosymplectic characters using lattice paths. A thorough treatment of the representation theory of Lie superalgebras can be found in [4]. In [2], a tableau description is given for characters of orthosymplectic Lie algebras spo(2m, n). These characters are shown to be hybrids of symplectic Schur functions and (general linear) Schur functions and can be described in terms of the spo(2m, n)-tableaux introduced in [2]. We focus on this combinatorial description to prove determinantal formulas.

Given two partitions \( \lambda \) and \( \mu \), we have \( \mu \subseteq \lambda \) if \( \mu_i \leq \lambda_i \) for \( i \geq 1 \). The skew diagram of shape \( \lambda/\mu \) is formed by removing the Young diagram of shape \( \mu \) from the upper left-hand corner of the Young diagram of shape \( \lambda \). In the special case where \( \mu = \emptyset \), the skew diagram of shape \( \lambda/\mu \) is just the Young diagram of shape \( \lambda \). A skew tableau of shape \( \lambda/\mu \) is a filling of the skew diagram of shape \( \lambda/\mu \) with entries from a given set. When the entries come from a set of positive integers, the weight of a skew tableau and the skew Schur polynomial, \( s_{\lambda/\mu}(x_1, \ldots, x_n) \), are defined as in Section 2.

Let \( m \) and \( n \) be fixed positive integers and let \( B_0 = \{1^\circ, 2^\circ, \ldots, m, \overline{m}^\circ\} \), \( B_1 = \{1^\circ, 2^\circ, \ldots, n^\circ\} \) and \( B = B_0 \cup B_1 \). Order \( B \) as follows:

\[
1 < \overline{1} < 2 < \overline{2} < \ldots < m < \overline{m} < 1^\circ < 2^\circ < \cdots < n^\circ.
\]

An spo(2m, n)-tableau \( T \) of shape \( \lambda \) is a filling of the Young diagram of shape \( \lambda \) with entries from \( B \) that satisfies the following conditions:

1. the subtableau \( S \) of \( T \) that contains entries only from \( B_0 \) is semistandard symplectic;
(2) the skew tableau formed by removing $S$ from $T$ is strictly increasing across
the rows from left to right and weakly increasing down columns from top
to bottom.

Let $\mathcal{X} = \{x_1, x_1^{-1}, \ldots, x_m, x_m^{-1}\}$, $Y = \{y_1, \ldots, y_n\}$ and $Z = \mathcal{X} \cup Y$ and suppose
that $\ell(\lambda) \leq m$. In [2, Theorem 4.24], several equivalent definitions are given for
orthosymplectic characters, one of which is as follows:

$$
sp_{\lambda}(Z) = \sum_{\mu \subseteq \lambda} sp_{\mu}(\mathcal{X}) s_{\lambda \setminus \mu}(Y),
$$

where $sp_{\mu}(\mathcal{X})$ is a symplectic Schur polynomial in the variables $\mathcal{X}$ and $s_{\lambda \setminus \mu}(Y)$
is the usual skew Schur polynomial in the variables $Y$.

Given an $spo(2m, n)$-tableau $T$, replace each $i$ in $T$ with $x_i$, each $\bar{i}$ in $T$ with
$x_i^{-1}$ and each $i^o$ in $T$ with $y_i$. Define $wt(T)$ to be the product of these variables. Then ([2, Theorem 5.1]) $spo_{\lambda}(Z) = \sum_T wt(T)$, where the sum is over the
$spo(2m, n)$-tableaux of shape $\lambda$ with entries from $Z$.

**Example 4.1.** The tableau $T = \begin{array}{ccc}
1 & \bar{1}^o & \bar{1}^o \\
2 & \bar{1}^o & 3^o \\
1^o & 2^o & 3^o \\
1^o & 2^o
\end{array}$
is an $spo(8, 3)$-tableau of shape

$\lambda = (4, 3, 3, 2)$ and $wt(T) = x_1^{-1}x_2x_4^{-1}y_1^3y_2^2y_3^2$.

We will give lattice path proofs of Jacobi-Trudi formulas for orthosymplectic
Schur polynomials using a similar approach to that used in the previous section.
Along the $x$-axis, label the lines $x = 1$, $x = 2$, up to $x = 2m$ consecutively with the
symbols $1, \bar{1}, 2, \bar{2}, \ldots, m, \bar{m}$. Label the lines $x = 2m + 1$, $x = 2m + 2$, et cetera, up
to $x = 2m + n$ consecutively with the symbols $1^o, 2^o, \ldots, n^o$. There is a bijection between $spo(2m, n)$-tableaux with $\ell(\lambda) \leq m$ and families of paths $(\pi_1, \pi_2, \ldots, \pi_k)$,
where each $\pi_i$ has initial and terminal points given by $P_i = (2i - 1, k - i)$ and
$Q_i = (2m + n + 1, \lambda_i + k - i)$ respectively and steps restricted to $(1, 0)$ and $(0, 1)$
on $\{(x, y) \mid 1 \leq x \leq 2m\}$ and $(1, 0)$ and $(1, 1)$ on $\{(x, y) \mid 2m + 1 \leq x \leq 2m + n\}$. Associate the $i$th row of an $spo(2m, n)$-tableau $T$ with shape $\lambda = (\lambda_1, \ldots, \lambda_k)$ to a
path $\pi_i$ with initial point $P_i$ by placing a vertical step at $a$ (respectively $\bar{a}$) for each occurrence of $a$ in the $i$th row of $T$ for $1 \leq a \leq m$ and place a diagonal step starting
at $a^o$ if $a^o$ appears in the $i$th row of $T$. Fill in the remainder of the path with
horizontal steps. Given a path $\pi$, assign a weight $x_{a}$ to each vertical step at $a$, $x_{a}^{-1}$
to each vertical step at $\bar{a}$ and $y_{a}$ to each diagonal step above $a^o$. Assign a weight
of one to every horizontal step in the path and define $wt(\pi)$ to be the product of the
weights of each of the steps. For a family of paths $\epsilon = (\pi_1, \pi_2, \ldots, \pi_k)$, define
$wt(\epsilon)$ to be the product of the weights $wt(\pi_i)$ and define $sgn(\epsilon)$ as in (1).

**Example 4.2.** The family of lattice paths corresponding to the rows of the
tableau $T$ of Example 4.1 is given below.
If $\pi_i$ denotes the path from $P_i$ to $Q_i$ and $\epsilon = (\pi_1, \pi_2, \pi_3, \pi_4)$, then
$\text{wt}(\epsilon) = x_1 y_1^2 y_2^2 y_3^2$.

For $i \geq 1$, define $H^{(i)}_r = \sum_{\ell=0}^r h_\ell(x_i, x_i^{-1}, \ldots, x_m, x_m^{-1})e_{r-\ell}(Y)$ and $H_r(Z) = H^{(1)}_r$.

**Theorem 4.3.** Let $\lambda$ be a partition with $\ell(\lambda) = k \leq m$. Then
$\text{spo}_\lambda(Z) = |H^{(i)}_{\lambda_j-j+i}|$.

**Proof.** Let $P_i = (2i-1, k-i)$ and $Q_j = (2m+n+1, \lambda_j+k-j)$ and consider paths from $P_i$ to $Q_j$ with steps restricted to $(1,0)$ and $(0,1)$ on $\{(x,y) \mid 1 \leq x \leq 2m\}$ and to $(1,0)$ and $(1,1)$ on $\{(x,y) \mid 2m+1 \leq x \leq 2m+n\}$. Then
$$\sum_{P_i \to Q_j} \text{wt}(\pi) = \sum_{\ell=0}^r h_\ell(x_i, x_i^{-1}, \ldots, x_m, x_m^{-1})e_{\lambda_j-j+i-\ell}(Y),$$
where the sum runs over all such paths $\pi$ with initial and terminal points given by the $P_i$ and $Q_j$ so
$$|H^{(i)}_{\lambda_j-j+i}| = \sum_\epsilon (-1)^\epsilon \text{wt}(\epsilon),$$
where the sum is over all families of paths $\epsilon = (\pi_1, \ldots, \pi_k)$ for which $\pi_i$ has initial point $P_i$ and terminal point $Q_j$ for some unique $j$, $1 \leq i, j \leq k$. As in the proof of Theorem 3.2, it can be shown that the summands on the right-hand side of the above equation that correspond to intersecting families cancel in pairs, so that the sum is over all nonintersecting families. Since there is a bijection between such families and $\text{spo}(2m,n)$-tableaux, the result follows. $\square$

The following Lemma allows us to obtain Corollary 4.5 – an orthosymplectic Jacobi-Trudi identity that is derived algebraically in [2, Theorem 4.24 e)].

**Lemma 4.4.** Let $p \geq 1$ and let $A = (H^{(p+i+1)}_{r_j+i-1})_{1 \leq i, j \leq n}$. Then
$$|A| = |H^{(p)}_{r_j} : H^{(p)}_{r_j+i-1} + H^{(p)}_{r_j-(i-1)}|,$$
where $H^{(p)}_{r_j}$ gives the first row of the matrix for $1 \leq j \leq n$ and subsequent rows are given by $H^{(p)}_{r_j+i-1} + H^{(p)}_{r_j-(i-1)}$ for $i \geq 2$. 
Proof. Using Lemma 3.3, we have 
\[ H_{r}^{(p+1)} = H_{r}^{(p)} + (x_{p} + x_{p}^{-1})H_{r-1}^{(p)} + H_{r-2}^{(p)}, \]
for \( p \geq 1 \). The theorem now follows using a proof similar to that used to prove Theorem 3.4. \( \square \)

**Corollary 4.5.** Let \( \lambda \) be a partition with \( \ell(\lambda) \leq m \). Then

\[ spo_{\lambda}(Z) = |H_{\lambda_{j}-j+1}(Z) : H_{\lambda_{j}-j+i}(Z) + H_{\lambda_{j}-j-i+2}(Z)|. \]

To associate columns of \( spo(2m,n) \)-tableaux with lattice paths, we consider paths with initial points given by \( P_{i} = (0, -i) \) and terminal points given by \( Q_{j} = (2m + n - 1, \lambda_{j} - j), 1 \leq i, j \leq \lambda_{1} \), and steps restricted to \((1,0)\) and \((0,1)\) in the region \( \{(x, y) | x \leq 2m - 1\} \) and to \((1,0)\) and \((0,1)\) in the region \( \{(x, y) | x \geq 2m\} \). There is a bijection between \( spo(m,n) \)-tableaux with \( \ell(\lambda) \leq m \) and families of paths \((\pi_{1}, \pi_{2}, \ldots, \pi_{\lambda_{1}})\), where \( \pi_{i} \) has initial point \( P_{i} \) and terminal point \( Q_{i} \), steps described as above, and such that no step of \( \pi_{1} \) starting at a point with \( x \)-coordinate less than \( 2m \) intersects the line \( y = \frac{1}{2}x \). Note that while the diagonal steps of \( \pi_{1} \) do not intersect the line \( y = \frac{1}{2}x \), it is possible for vertical steps to intersect it.

Label the \( x \)-axis from \( x = 0 \) to \( x = 2m - 1 \) with the symbols \( 1, \overline{1} \) et cetera, up to \( m, \overline{m} \) and label the lines \( x = 2m \) to \( x = 2m + n - 1 \) with \( 1^{\circ}, \ldots, n^{\circ} \). Associate the \( i \)th column of an \( spo(2m,n) \)-tableau with the path \( \pi_{i} \) from \( P_{i} \) to \( Q_{i} \) that has a diagonal step starting at \( a \) (respectively \( \overline{a} \)) if an \( a \) appears in the \( i \)th column of \( T \), and a vertical step at \( a^{\circ} \) for each occurrence of \( a^{\circ} \) in the \( i \)th column of \( T \). Horizontal steps can be placed in a unique way to connect the path. The argument justifying that \( T \) violates the symplectic condition if and only if a diagonal step in the path associated to the first column of \( T \) intersects \( y = \frac{1}{2}x \) at a point \( A \) is similar to that presented for symplectic tableaux in Section 3. The weight of a given path between \( P_{i} \) and \( Q_{j} \) and the weight of a family of paths is as defined for the previous case.

**Example 4.6.** The family of paths associated to the columns of the tableau \( T \) from Example 4.1 is as follows.
To prove the dual Jacobi-Trudi formula, we define the reflection of a path $\pi_i$ from $P_i$ to $Q_j$, that first intersects $y = \frac{1}{2}x$ with a diagonal step ending at $A = (x, y)$, for $x \leq 2m$, to be the path $\pi'_i$ from $P'_i = (0, i)$ to $Q_j$ where the steps of $\pi'$ west of $A$ are as described in Section 3 and the steps of $\pi'$ east of $A$ are the same as those in $\pi_i$. Let $Q$ denote the set of families of paths $\epsilon = (\pi_1, \ldots, \pi_{\lambda_1})$, where $\pi_i$ is a path from $P_i$ to $Q_i$ with steps starting in the region $\{(x, y) \mid x \leq 2m - 1\}$ restricted to $(1, 0)$ and $(1, 1)$ and steps starting in the region $\{(x, y) \mid x \geq 2m\}$ restricted to $(1, 0)$ and $(0, 1)$, and such that no diagonal step in $\pi_1$ intersects $y = \frac{1}{2}x$. Let $Q'$ denote the families of paths $\epsilon' = (\pi'_1, \ldots, \pi'_{\lambda_1})$, where $\pi'_i$ has initial point $P'_i$ and terminal point $Q_i$, with steps starting in the region $\{(x, y) \mid x \leq 2m - 1\}$ restricted to $(1, 0)$ and $(1, 1)$ and steps starting in the region $\{(x, y) \mid x \geq 2m\}$ restricted to $(1, 0)$ and $(0, 1)$. It follows that reflecting paths in the way described above gives a weight-preserving bijection between $Q$ and $Q'$.

Example 4.7. Below is the reflection of a path that intersects $y = \frac{1}{2}x$.

Define $E_r(Z) = \sum_{\ell=0}^r e_{\ell}(X) h_{r-\ell}(Y)$. The following theorem, which is proved algebraically in [2, Theorem 4.24 (g)] can be proved using our weight-preserving bijection between $Q$ and $Q'$ and a proof analogous to that of the proof of Theorem 3.8.

Theorem 4.8. Let $\lambda$ be a partition with $\ell(\lambda) = k \leq m$. Then

$$\text{spo}_\lambda(Z) = |E_{\lambda'_j - i}(Z) - E_{\lambda'_j - j}(Z)|.$$ 

To prove an orthosymplectic Giambelli determinantal identity, we will work with paths comprised of a northwesterly portion, where all steps terminate at a point with $y$-coordinate less than or equal to $-1$ and a northeasterly portion, where all steps terminate at a point with $y$-coordinate greater than $-1$. Label the lines $x = 0, x = 1, \ldots, x = 2m - 1$ consecutively with the symbols $1, 1, \ldots, m, m$ and label the lines $x = 2m, x = 2m + 1, \ldots, x = 2m + n - 1$ consecutively with the symbols $1^\circ, 2^\circ, \ldots, n^\circ$. In the northwesterly portion, steps starting at a point with $x$-coordinate label $a^\circ$ are restricted to $(-1, 0)$ and $(-1, 1)$, and steps starting at a point with $x$-coordinate label $a$ or $\overline{a}$ are restricted to $(-1, 0)$ and $(0, 1)$. In the northeasterly portion, steps starting at a point with $x$-coordinate label $a$ or $\overline{a}$ are
restricted to \((1, 0)\) and \((1, 1)\), and steps starting at a point with \(x\)-coordinate label \(a^\circ\) are restricted to \((1, 0)\) and \((0, 1)\).

Let \(\lambda = (\alpha_1, \ldots, \alpha_r | \beta_1, \ldots, \beta_r)\) be the Frobenius notation for \(\lambda\) and let \(P_i = (2m + n - 1, -1 - \alpha_i)\) and \(Q_i = (2m + n - 1, \beta_i)\), for \(1 \leq i \leq r\). We will associate each of the \(i\)th principal hooks of an \(\text{spo}(2m, n)\)-tableau \(T\) with \(\ell(\lambda) \leq m\) to a path \(\pi_i\) from \(P_i\) to \(Q_i\) with steps having the above restrictions. Let \(a\) denote the entry in the \((i, i)\)-box of \(T\). The northwesterly portion of \(\pi_i\) will be a path from \(P_i\) to the point with \(x\)-coordinate label \(a\) (or \(a^\circ\)) and \(y\)-coordinate \(-1\), which we will call \(R_i\). We form this path by placing a diagonal step pointed west starting at each \(b \in \{1^\circ, 1, \ldots, n^\circ\}\) that appears to the right of the \((i, i)\)-box in the \(i\)th principal hook of \(T\) and a vertical step on the line with \(x\)-coordinate label \(b \in \{1, 1^\circ, \ldots, m, m^\circ\}\) that appears to the right of the \((i, i)\)-box in the \(i\)th principal hook of \(T\). There will necessarily be either a diagonal or vertical step starting at \(R_i\) and there is a unique way to complete \(\pi_i\) with horizontal steps, directed west. This gives a bijection between the set of \(\text{spo}(2m, n)\)-tableaux and the set of families \(\epsilon = (\pi_1, \pi_2, \ldots, \pi_r)\) where each path \(\pi_i\) has initial point \(P_i\), terminal point \(Q_i\), step restrictions as described above, and \(\pi_1\) does not intersect the line \(y = \frac{1}{2}x\).

**Example 4.9.** Consider the \(\text{spo}(8, 3)\)-tableau \(T\) from Example 4.1. The Frobenius notation for \(\lambda\) is \(\lambda = (3, 1, 0 | 3, 2, 0)\) and \(T\) has rank three. The family of paths \(\epsilon = (\pi_1, \pi_2, \pi_3)\) associated to the principal hooks of \(T\) is as follows.

![Diagram](image)

Following the reasoning used in the proof of Theorem 3.10, we obtain the following theorem.

**Theorem 4.10.** Let \(\lambda = (\alpha | \beta)\) be the Frobenius notation for a partition \(\lambda\) with \(\ell(\lambda) \leq m\) and rank \(r\). Then

\[\text{spo}_\lambda(Z) = |\text{spo}_{(\alpha_i | \beta_j)}(Z)|_{1 \leq i, j \leq r}\]
References


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